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61st Annual UNSW School Mathematics Competition: Competition Problems and Solutions

Compiled by Denis Potapov[1]

A Junior Division – Problems

Problem A1:
Alice plays the following game. She writes every number from 1 to 125 in her book. On every move, she replaces a couple of numbers with the remainder after dividing the sum of these numbers by 11. What number will be in the book after 124 moves?

Problem A2:
Prove that the least common multiple of set $1, 2, \dots, 2n$ and the least common multiple of set $n+1, n+2, \dots, 2n$ are equal.

Problem A3:
Two players play the following game on a chess board. On every move, a player adds a bishop to the board. Disregarding the colour of the added pieces, a player can only put the bishop in the square that is not under attack by the previously added bishops. The player that cannot make a move loses. Find the winner and the winning strategy.

Problem A4:
There are three groups of octopuses in a fish tank. Each group has its colour. The number of octopuses in each group is 13, 15, and 17. When two octopuses of different colours approach each other, they change colour to the third one. Can a uniform colour be achieved?

Problem A5:
Thirty members of the parliament (MP) sitting on five parliamentary commissions proposed forty amendments to legislation. Each MP proposed at least one amendment, and no two MPs proposed the same one. Any two MPs on the same commission proposed the same number of amendments, and any two on different commissions proposed different numbers. How many MPs proposed exactly one amendment?

Problem A6:
Prove that the following triangles congruence test is incorrect: The triangles ABC and $A_1B_1C_1$ are congruent if sides AB and A_1B_1 are equal ($AB = A_1B_1$), sides AC and A_1C_1 are equal ($AC = A_1C_1$), and angles C and C_1 are equal ($\angle BCA = \angle B_1C_1A_1$).

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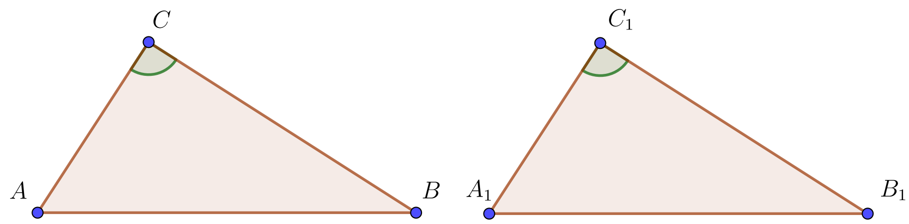


C
A
B
C_1
A_1
B_1

B Senior Division – Problems

Problem B1:
Two players play the following game on a chess board. On every move, a player adds a bishop to the board. Disregarding the colour of the added pieces, a player can only put the bishop in the square that is not under attack by the previously added bishops. The player that cannot make a move loses. Find the winner and the winning strategy.

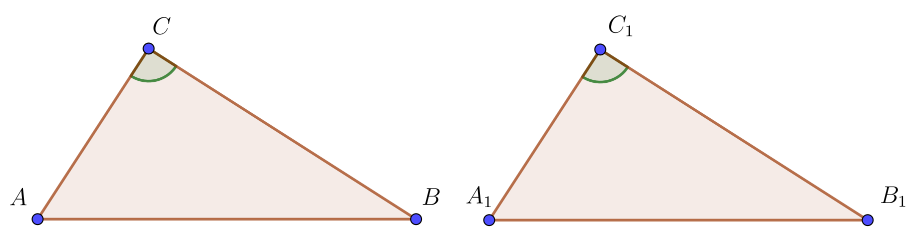
Problem B2:
Prove that the product of all the divisors of a positive integer n equals $\sqrt{n^s}$ where s is the number of divisors.

Problem B3:
Bob writes the numbers $1, 2, \dots, 2023$ on the board. On each move, he erases two numbers and writes the sum of the erased numbers minus 2023 instead. What will be the numbers on the board after 2022 moves?

Problem B4:
Sixty members of parliament (MP) attend a parliamentary commission. Among any ten MPs, there are three from the same party. Prove that at least fifteen MPs on the commission are from the same party.

Problem B5:
The following triangles congruence test is incorrect. Claim: Triangles ABC and $A_1B_1C_1$ are congruent if the sides AB and A_1B_1 are equal ($AB = A_1B_1$), the sides AC and A_1C_1 are equal ($AC = A_1C_1$), and the angles C and C_1 are equal ($\angle BCA = \angle B_1C_1A_1$).

Inspect the following quasi-proof of this claim and find the error.

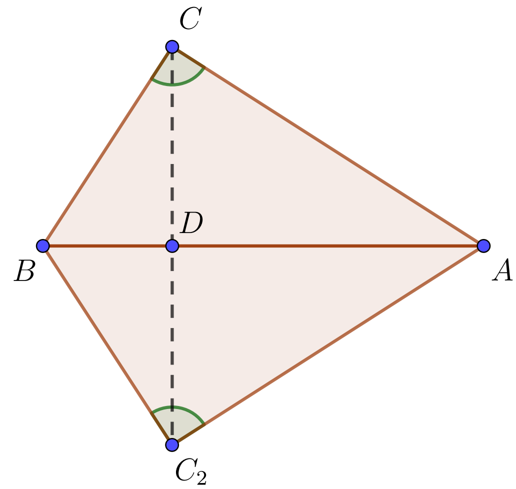


Proof. [Quasi-Proof]

As shown in the diagram below[2], we construct the triangle ABC_2 to equal $A_1B_1C_1$. We then draw the line CC_2. Since we have the sides AC and AC_2 equal, the triangle ACC_2 is isosceles. Hence, the angles ACC_2 and AC_2C are also equal. By construction, the angles ACB and AC_2B are equal. Therefore, the angles BCC_2 and BC_2C are also equal; hence the triangle BCC_2 is also isosceles. Accordingly, the sides BC

[2]Due to a technical issue, the printout of the problem sent to schools did not include the diagram. The missing diagram made the correct problem interpretation difficult.

and BC_2 are equal. Now, since the sides AC and AC_2 equal, the sides BC and BC_2 equal, and the angles ACB and AC_2B equal, the triangles ACB and AC_2B are congruent by the side-angle-side criterion. The (quasi-)proof is complete.



□

A Junior Division – Solutions

Solution A1.

The answer is the remainder after dividing the sum of all the numbers from 1 to 125 by 11:

$$1 + 2 + \cdots + 125 \mod 11 = \frac{1+125}{2} \times 125 \mod 11 = 10\,.$$

□

Solution A2.

Let $S = \{1, 2, \ldots, 2n\}$ and $T = \{n+1, n+2, \ldots, 2n\}$. Let $a = \operatorname{lcm} S$ and $b = \operatorname{lcm} T$. Since the number a is a common multiple for the set T, we have $a = mb$ for some $m \in \mathbb{Z}$.

Let's show that the number b is a common multiple for the set S. That is, let show that the number b is a common multiple for the set $S_1 = \{1, 2, \ldots, n\}$.

Take $1 \le k \le n$. Let's show that the number b is a multiple of k. We prove this claim in two steps. First, if $k = n$, then the number b is a multiple of k. Second, if $1 \le k \le n - 1$, then, since

$$\frac{2n - (n+1)}{k} = \frac{n-1}{k} \ge 1\,,$$

there is a integer s such that

$$\frac{n+1}{k} \le s \le \frac{2n}{k}\,.$$

That proves the claim.

Hence, we conclude that the number b is a common multiple for the set S and therefore $a = b$. □

Solution A3.

The second player wins. We base our strategy on symmetry with respect to the horizontal centre line. On every move, the second player puts their bishop in the square symmetric to the previous move of the first player. With such a strategy, the bishop's position is symmetric after every move of the second player. If a move is available for the first player, then there is also a (symmetric) move for the second player. □

Solution A4.

We denote the number of octopuses in each group x, y and z. On each interaction, one of the following changes to the triple (x,y,z) happens:

$$\begin{aligned}
(x,y,z) &\to (x-1,y-1,z+2) \\
(x,y,z) &\to (x-1,y+2,z-1) \\
(x,y,z) &\to (x+2,y-1,z-1)
\end{aligned}$$

For each change above, we have:

$$\begin{aligned}
x-y &\equiv 0 \mod 3 \\
x-z &\equiv 0 \mod 3 \\
y-z &\equiv 0 \mod 3
\end{aligned}$$

On the other hand, at the beginning, we have

$$x = 13, \quad y = 15, \quad z = 17$$

and therefore

$$\begin{aligned}
x-y &\equiv 1 \mod 3 \\
x-z &\equiv 2 \mod 3 \\
y-z &\equiv 1 \mod 3
\end{aligned}$$

Hence, the final state, where, i.e., $x = y = 0$ is not achievable. □

Solution A5.

Answer: 26.

Assume that $n_1, n_2, \ldots, n_5$ is the number of MPs on each of the commissions and $a_1 = 1 < a_2 < \ldots < a_5$ is the number of amendments proposed by a member of the corresponding commission. We then have

$$\begin{aligned}
n_1 + n_2 + n_3 + n_4 + n_5 &= 30 \\
n_1 + a_2 n_2 + a_3 n_3 + a_4 n_4 + a_5 n_5 &= 40
\end{aligned}$$

Subtracting, we have

$$(a_2 - 1)n_2 + (a_3 - 1)n_3 + (a_4 - 1)n_4 + (a_5 - 1)n_5 = 10\,.$$

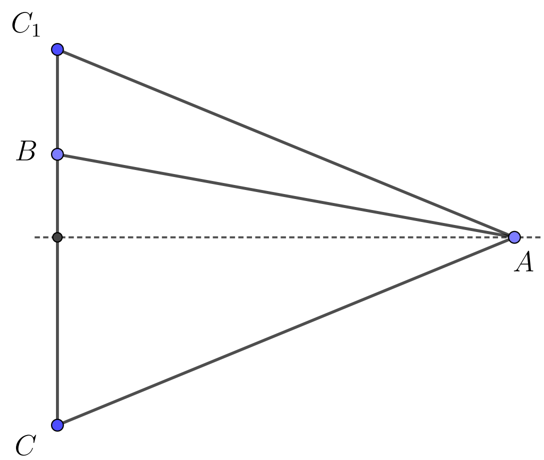
Since $n_j \geq 1$ and $2 \leq a_2 < a_3 < a_4 < a_5$, the only solution to the latter equation is

$$n_2 = \cdots = n_5 = 1 \quad \text{and} \quad a_2 = 2,\ a_3 = 3,\ a_4 = 4,\ a_5 = 5\,.$$

Hence, $n_1 = 26$. □

Solution A6.

We prove that the claim is incorrect by providing a counter-example. We construct the example as follows. We start with isosceles triangle $\triangle ACC_1$ with sides $AC = AC_1$. Then we choose a point B on the base CC_1 strictly between the point C and the middle of the base CC_1. The triangles $\triangle ABC$ and $\triangle ABC_1$ are not congruent but satisfy the conditions of the claim.



□

B Senior Division – Solutions

Solution B1.

The second player wins. We base our strategy on symmetry with respect to the horizontal centre line. On every move, the second player puts their bishop in the square symmetric to the previous move of the first player. With such a strategy, the bishop's position is symmetric after every move of the second player. If a move is available for the first player, then there is also a (symmetric) move for the second player. □

Solution B2.

If q_i is a divisor of the number n, then $n = q_i p_i$ and the number p_i is also a divisor. If $q_1 < q_2 < \cdots < q_s$ is the divisors in increasing order, then $p_1 > p_2 > \cdots > p_s$ are the same divisors in the decreasing order. So, if

$$S = q_1 \times \cdots \times q_s = p_1 \times \cdots \times p_s \,,$$

then

$$S^2 = (q_1 p_1) \times \cdots \times (q_s p_s) = n^s \,.$$

Therefore, the product of all the divisors of a positive integer n is $S = \sqrt{n^s}$, as required. □

Solution B3.

For n numbers on the board, we compute the following invariant:

$$I = x_1 + \cdots + x_n - n \times 2023 \,.$$

On every move, any two numbers a and b are replaced by $a + b - 2023$, i.e.,

$$(a, b) \to a + b - 2023 \,.$$

The value I is indeed an invariant: if S is the sum of the numbers on the board except a and b, then the value of the invariant before the move is

$$I_1 = S + a + b - n \times 2023 \,,$$

and the value after the move is

$$I_2 S + a + b - 2023 - \big(n - 1\big) \times 2023 \,.$$

Since $I_1 = I_2$, the value is indeed an invariant.

The value of the invariant at the beginning of the process is

$$\begin{aligned} 1 + 2 + \cdots + 2023 - 2023 \times 2023 &= \frac{1 + 2023}{2} \times 2023 - 2023 \times 2023 \\ &= 2023 \times \left(\frac{1 + 2023}{2} - 2023\right) \\ &= 2023 \times (-1011) \,. \end{aligned}$$

Now, note that on every move, two numbers are removed and one number is added. Thus, after 2022 moves, there is only one number written on the board. Using the invariant value for the case when there is only one number on the board, we find this number:

$$\begin{aligned}\text{remaining number} &= -2023 \times 1011 + 1 \times 2023 \\ &= -2023 \times 1010\,.\end{aligned}$$

□

Solution B4.

We will prove the claim by contradiction. Assume that each party has at most fourteen MPs on the commission (the contrary assumption). We will show such assumption leads to a contradiction.

Let n_1, n_2 and n_3 be the number of parties that sent one MP (group A), two MPs (group B) and at least three (and at most fourteen) MPs (group C), respectively. Then

$$n_1 + 2n_2 + 14n_3 \geq 60 \quad \Leftrightarrow \quad n_3 \geq \frac{60 - n_1 - 2n_2}{14}\,.$$

Let's choose all of the MPs in the group A and the group B and choose two MPs from each of the party in the group C. The total number of MPs chosen is

$$N = n_1 + 2n_2 + 2n_3\,.$$

According to the condition of the problem that every group of ten MPs has at least three from the same party, the value N is less than ten. Let's show that the contrary assumption implies that N is at least ten. If we do so, then we will obtain the contradiction and the proof will be finished.

We will show that $N \geq 10$ using two mutually exclusive cases. First, assume that $n_1 = n_2 = 0$. In such case,

$$n_3 \geq \frac{60 - n_1 - 2n_2}{14} = \frac{30}{7} > 4$$

so $n_3 \geq 5$. Hence,

$$N = n_1 + 2n_2 + 2n_3 \geq 2n_3 \geq 10\,.$$

Second, assume that $n_1 + 2n_2 \geq 1$. In such case, we have

$$N = n_1 + 2n_2 + 2n_3 \geq n_1 + 2n_2 + \left(\frac{60}{7} - \frac{n_1}{7} - \frac{2n_2}{7}\right) \geq \frac{6}{7}(n_1 + 2n_2) + \frac{60}{7} \geq \frac{6}{7} + \frac{60}{7}\,;$$

therefore, $N \geq 10$. So, in both cases, we have $N \geq 10$. □

Alternative Solution B4.

Define Condition (*) as the condition that there are three MPs from the same party among any set of ten MPs.

Now, for a contradiction, suppose that there is no party that has more than fourteen MPs on the commission.

Suppose that n_1 is the number of Type I parties which are the parties with exactly one MP on the commission and suppose that n_2 is the number of Type II parties which are all the other parties (i.e., parties that have at least two members on the commission). We will proceed by considering cases on n_1.

If $n_1 \leq 3$, then there are at least $60 - 3 = 57$ MPs that belong to a Type II party. Since $57 > 14 \times 4$, $n_2 \geq 5$ by the Pigeonhole Principle. Selecting two MPs from each of any five of the Type II parties gives a set of ten MPs that contradicts Condition (*).

If $4 \leq n_1 \leq 9$, then there are at least $60 - 9 = 51$ MPs that belong to a Type II party. Since $51 > 14 \times 3$, $n_2 \geq 4$ by the Pigeonhole Principle. Selecting one MP from each of any two of the Type I parties and selecting two MPs from each of any four of the Type II parties gives a set of ten MPs that contradicts Condition (*).

If $n_1 \geq 10$, then selecting one MP from each of any ten Type I parties gives a set of ten MPs that contradicts Condition (*).

Hence, by exhaustion of cases, we have shown that there must be at least fifteen MPs on the commission from the same party. $\square$

Second Alternative Solution B4 (by Jason Phan from James Ruse Agricultural High School, re-written by Yudhi Bunjamin).

We define a *group* to be a subset of MPs that are known to belong to the same party. Then any set of 10 MPs must contain a group of size 3. The goal is to construct a group of size at least 15.

Consider any set of 10 MPs and suppose this gives us the group G_1. Next, consider a set of 10 of the $60 - |G_1| = 57$ MPs that do not belong G_1. This gives us the group G_2 which is disjoint from G_1 and $57 - |G_2| = 54$ MPs that do not belong to any group yet. We can repeat this process until the number of MPs that do not belong to any group is less than 10. This means that we are able to get $(60 - 9)/3 = 17$ groups $G_1, G_2, \ldots, G_{17}$ which are all of size 3 and are all pairwise disjoint.

Now, consider a set of 10 MPs consisting of one MP from each of $G_1, G_2, \ldots G_{10}$. Then three of these MPs must form a group. Without loss of generality, suppose that this group consists of the MPs from G_1, G_2 and G_3. This implies that the set of MPs $H_1 = G_1 \cup G_2 \cup G_3$ forms a group. Next, consider a set of 10 MPs consisting of one MP from each of $G_4, G_5, \ldots G_{13}$ and without loss of generality, suppose that the MPs from G_4, G_5 and G_6 from a group. Then this means that the set of MPs $H_2 = G_4 \cup G_5 \cup G_6$ forms a group. We can this repeat this process one more time and this gives, without loss of generality, a group $H_3 = G_7 \cup G_8 \cup G_9$. This now gives us the groups H_1, H_2 and H_3 which are all of size nine. Note that the groups $H_1, H_2, H_3, G_{10}, G_{11}, \ldots, G_{16}$ are all pairwise disjoint.

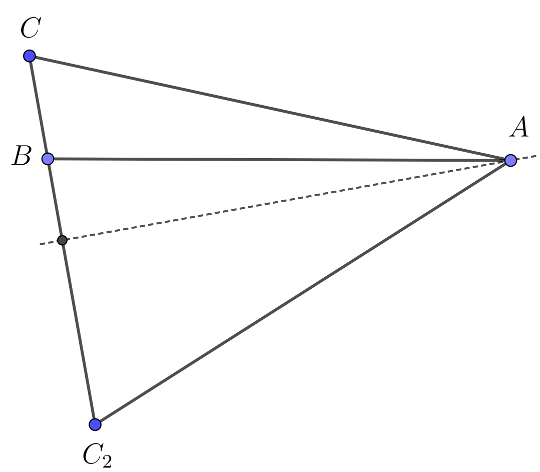
Next, a consider a set of 10 MPs consisting of one MP from each of H_1, H_2, H_3, G_{10}, $G_{11}, \ldots, G_{16}$. Then three of these MPs must come from the same party. If any of these three MPs belong to H_1, H_2 or H_3, then this implies that we can form a group of size at least $9 + 3 + 3 = 16$ so we are done. Otherwise, this implies that three MPs, each belonging to one of $G_{10}, G_{11}, \ldots, G_{17}$, form a group so this give us another group of size 9. Without loss of generality, suppose that this group is $H_4 = G_{10} \cup G_{11} \cup G_{12}$.

Now, consider a set of 10 MPs consisting of two MPs from each of G_{13}, G_{14}, G_{15}, G_{16}, G_{17}. Then the three MPs in this set that come from the same party must belong to at least two of G_{13}, G_{14}, G_{15}, G_{16}, G_{17}. Thus, without loss of generality, this gives us a group $J_1 = G_{13} \cup G_{14}$ consisting of 6 MPs. We now have the pairwise disjoint groups H_1, H_2, H_3, H_4 and J_1.

Finally, consider a set of 10 MPs consisting of two MPs from each of H_1, H_2, H_3, H_4 and J_1. Then the three MPs in this set that come from the same party must belong to at least two of H_1, H_2, H_3, H_4, J_1. This implies that we can form a final group of size at least $9 + 6 = 15$, as required. □

Solution B5.

The error in the proof is in the assumption that the point D ends up strictly between the points A and B. If the point D coincides with the point B, then the argument fails to show $BC = BC_2$. The reader can see this in the following counter-example.



The triangle $\triangle ACC_2$ is isosceles with sides $AC = AC_2$. The point B (and the point D) is chosen on the base CC_2 off the middle point. □