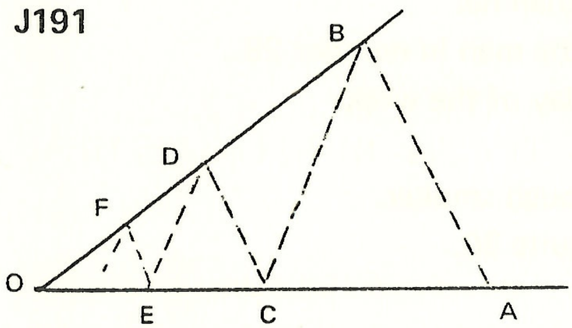
SOLUTIONS

Solutions to Problems 191–200 in Vol 8 No 3

The names of successful problem solvers appear after the solution to Problem 200.

Junior

J191



A yacht starts from a point A, one mile due East of a buoy O, and tacks up to the buoy as indicated in the diagram. First, it sails on the course AB, 30° West of North, until the buoy bears exactly South-West; then it tacks on the course BC, 30° West of South until the buoy's bearing is again due West. This process is repeated indefinitely as indicated by the broken line ABCDEFG. . . . After reaching the buoy, one of the deck hands (who wishes to take his mind off the incredible exertions of the last few feet of the trip) calculates how far they have actually sailed. Can you?

Answer Since all angles in ABC are 60° the triangle is equilateral, and the distance actually sailed (*AB + *BC) is exactly twice the distance travelled due west (*AC). The same argument applies for subsequent sections of the trip. Hence to travel west a total of 1 mile the distance actually sailed is 2 miles.

J192 Replace each E by an even digit (0, 2, 4, 6 or 8) and each O by an odd digit (1, 3, 5, 7 or 9) so that the following multiplication is correct.

$$\begin{array}{r} E\,E\,O \\ \underline{O\,O} \\ E\,O\,E\,O \\ \underline{E\,O\,O} \\ \underline{\underline{O\,O\,O\,O\,O}} \end{array}$$

Answer

$$\begin{array}{r} 285 \\ \underline{39} \\ 2\,565 \\ \underline{8\,55} \\ \underline{11{,}115} \end{array}$$

The digits 3 and 9 of the multiplier may be deduced immediately from the relative sizes of the first ($>$ 2100) and second ($<$ 900) lines of working. The initial digits of all lines may then be deduced. Since 3 times the multiplicand exceeds 810, the even second digit can only be 8. Then the last digit must be 5, since 3 times this number has an odd middle digit.

J193 If $1 \leqslant a \leqslant 2$, simplify $\sqrt{[a + 2\sqrt{(a-1)}]} + \sqrt{[a - 2\sqrt{(a-1)}]}$.

Answer $a + 2\sqrt{(a-1)} = (a-1) + 2\sqrt{(a-1)} + 1 = [1 + \sqrt{(a-1)}]^2$.
Hence $\sqrt{[a + 2\sqrt{(a-1)}]} = 1 + \sqrt{(a-1)}$.
Similarly $a - 2\sqrt{(a-1)} = [1 - \sqrt{(a-1)}]^2$
so that $\sqrt{[a - 2\sqrt{(a-1)}]} = 1 - \sqrt{(a-1)}$ if $\sqrt{(a-1)} \leqslant 1$, i.e. if $a \leqslant 2$.
Hence $\sqrt{[a + 2\sqrt{(a-1)}]} + \sqrt{[a - 2\sqrt{(a-1)}]} = [1 + \sqrt{(a-1)}] + [1 - \sqrt{(a-1)}]$
$= 2$ if $1 \leqslant a \leqslant 2$.

O194 Twenty four coins are of the same size and external appearance; some are pure gold, and the rest are of a lighter alloy. Using at most thirteen weighings on a pan balance, how can we determine how many coins are gold?

Answer Arrange the coins into twelve pairs. Take the first pair and compare the two coins on this pan balance.

If they are of unequal weight, place both coins on the L.H. pan. Place each of the other pairs in turn on the R.H. pan. The behaviour of the balance will determine whether there are 0, 1, or 2 gold coins in each of these other 11 pairs. Thus after only 12 weighings the number of gold coins is known.

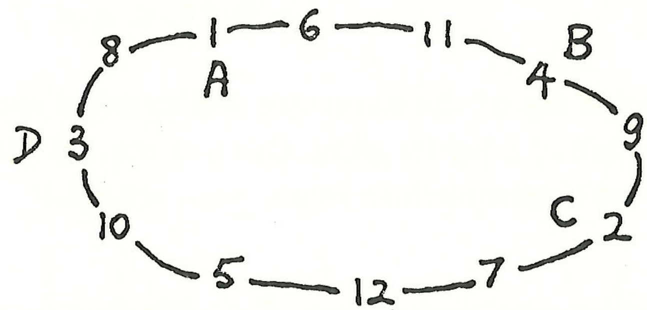
If the first pair are equal coins, they are again both placed on the L.H. pan, and the second pair on the R.H. pan. If a balance is obtained, all 4 coins are known to be of the same kind, and the second pair is replaced by the third pair. Eventually, since the coins are not all the same, a pair will be found which is either lighter than, or heavier than the first pair.

For definiteness, suppose that the first pair were heavier. Then all coins except those in the last pair used were gold. The coins of this last pair are now compared with each other. If they are of equal weight they are both alloy, otherwise the lighter one is alloy.

In either case we can choose one coin known to be gold, and one alloy and place them together on the L.H. pan. The remaining pairs, whose nature is not yet known, are now placed in turn on the R.H. pan to determine how many gold coins are present in each pair. This process uses 13 weighings in all.

O195 Four different buttons are placed on the numbers 1, 2, 3 and 4 of a horizontal clock face. A button may be moved in either a clockwise or an anticlockwise direction over four other numbers to a fifth number (i.e. it is moved 25 minutes one way or the other) provided its destination is not already occupied by another button. After a certain number of such moves the buttons again cover the numbers 1, 2, 3 and 4. How many different rearrangements of the four buttons are possible as a result of this process?

Answer Dudeney's "button and string" method of solution enables the problem to be easily solved. The clockface is imagined replaced by a loop of string with the



numbers in the order 1, 6, 11, 4, 9, 2, 7, 12, 5, 10, 3, 8; these being the destinations in order of a button starting at 1 and moving clockwise in accordance with the requirements of the problem. We start with the four buttons A, B, C and D on destinations 1, 4, 2 and 3 respectively. These buttons cannot pass one another since no two are allowed to occupy the same destination simultaneously. Thus their order round the loop of string never changes. Any one of them can be moved round to the destination 1; but once in position there is no further choice if the four destinations 1, 2, 3 and 4 are all filled. Hence there are only 4 possible arrangements.

O196 Prove that the sum of the same even power of nine consecutive integers, the first of which exceeds 1, cannot be any integral power of any integer.

Answer We show that the given sum is always divisible by 3, but never by 9, from which stage the stated result is obvious. The nine consecutive integers have remainders 0, 1, 2, 3, 4, 5, 6, 7, 8 on division by 9, in some order. The squares of the numbers have remainders 0, 1, 4, 0, 7, 7, 0, 4 and 1 when divided by 9, and the "2k-th" powers have remainders 0, 1, 4^k 0, 7^k 7^k, 0, 4^k and 1. The sum of these powers of the 9 consecutive integers thus leaves the same remainder as $2.(1 + 4^k + 7^k)$ when divided by 9. Since 4^3 and 7^3 both leave the remainder 1, $2(1 + 4^k + 7^k)$ and $2(1 + 4^{k+3} + 7^{k+3})$ leave the same remainders. Hence there are only 3 values of k to look at, viz k = 1, 2, and 3 for which the remainders are 6, 6, and 6. Thus the remainder is 6 for all values of k and the proof is completed.

0197 Does there exist a natural number n such that the fractional part of the number $(2 + \sqrt{2})^n$ exceeds 0.999999?

Answer Observe that

$$(2 + \sqrt{2})^n + (2 - \sqrt{2})^n$$

is a positive integer for all values of n. This may be seen by expanding both terms using the binomial theorem, when the terms involving odd powers of $\sqrt{2}$ appear with opposite signs, and so cancel out.

As $(2-\sqrt{2}) \approx .586 < 1$, for all sufficiently large values of n, $(2-\sqrt{2})^n <$ .000,001. (An easy exercise using logarithms to find how big n must be; the smallest value is 26). It follows that for all $n \geqslant 26$, $(2 + \sqrt{2})^n$ falls short of an integer by less than .000,001 so has a fractional part greater than .999,999.

0198 Prove that in the equation

$$N = N/2 + N/4 + N/8 + \ldots + N/2^n + \ldots$$

(where N is any natural number) every fraction may be replaced by the nearest whole number. (If the fraction ends in a ½, replace it, as usual, by the next whole number.)

Answer Express N in binary form as

$$N = a_k 2^k + a_{k-1} 2^{k-1} + \ldots + a_1 2 + a_0$$

where $a_k = 1$ and $a_j = 0$ or 1, if $j < k$.

Then $\frac{1}{2}N = a_k(2^{k-1} + a_{k-1}2^{k-2} + \ldots + a_1 + \frac{1}{2}a_0$ and the nearest integer is $a_k 2^{k-1} + a_{k-1}2^{k-2} + \ldots + a_1 + E$ where $E = 0$ if $a_0 = 0$ and $E = 1$ if $a_0 = 1$, i.e. $E = a_0$.

Similarly the nearest integers to N/2, N/4, N/8, N/16, ..., $N/2^k$, $N/2^{k+1}$, $N/2^{k+2}$ are

$a_k 2^{k-1} + a_{k-1}2^{k-2} + \ldots + a_2 2 + a_1 + a_0$

$a_k 2^{k-2} + a_{k-1}2^{k-3} + \ldots + a_2 + a_1$

$a_k 2^{k-3} + a_{k-1}2^{k-4} + \ldots + a_3 + a_2$

....

....

$a_k + a_{k-1}$

a_k

0

Adding, we obtain

$a_k(2^{k-1} + 2^{k-2} + \ldots + 2 + 1 + 1) + a_{k-1}(2^{k-2} + 2^{k-3} + \ldots + 2 + 1 + 1) + \ldots + a_2.4 + a_1.2 + a_0.$

Since $(2^{r-1} + 2^{r-2} + \ldots + 2 + 1) = 2^r - 1$, we easily simplify the sum to

$$a_k 2^k + a_{k-1}2^{k-1} + \ldots + a_2 2^2 + a_1 2 + a_0$$
$$= N.$$

0199 Prove that if the polynomial

$$P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1}x + a_n$$

assumes integral values for all integral values of x, then it is possible to express P(x) as

$$b_0 + b_1 x + b_2 \frac{x(x-1)}{1.2} + b_3 \frac{x(x-1)(x-2)}{1.2.3} + \ldots + b_n \frac{x(x-1)(x-2)\ldots(x-n+1)}{1.2.3\ldots n}$$

where b_0, b_1, ... b_n are integers.

Answer Let $F_r(x) = \frac{x(x-1)\ldots(x-r+1)}{1.2.\ldots.r}$

Then $F_r(x)$ is equal to the binomial coefficient ${}^xC_r$ when x is a positive integer greater than r, is equal to zero for $x = 0, 1, 2, \ldots r$ and is equal to ${}^{-x+r-1}C_r(-1)^r$ if x is a negative integer. Since the binomial coefficients are all integers $F_r(x)$ has the property of having integer values for all integer values of x.

Note next that any polynomial P(x) of degree n can be expressed (uniquely) in the form

$$P(x) = b_0 + b_1F_1(x) + b_2F_2(x) + \ldots + b_nF_n(x) \qquad (1)$$

Indeed since $F_r(x)$ is a polynomial of degree r, the only term in x^n on the R.H.S. is $b_n.x^n/n!$ occurring in $b_nF_n(x)$ so that b_n must be taken equal to n! times the coefficient of x^n in P(x). Then, after subtracting $b_nF_n(x)$ from P(x) we are left with a polynomial of degree (n–1), which enables us to determine b_{n-1}, and so on.

Another way of obtaining the coefficients b_j recursively is as follows.

Set x = 0 in (1). We obtain $P(0) = b_0$ (since $F_r(0) = 0$). Now set x = 1 obtaining $P(1) = b_0 + b_1$ since $F_1(1) = 1$, $F_r(1) = 0$ for $r > 1$.

Hence $b_1 = P(1) - b_0$. Similarly setting $x = 2, 3, \ldots m$ we obtain

$b_2 = P(2) - b_0 - b_1F_1(2)$

$b_3 = P(3) - b_0 - b_1F_1(3) - b_2F_2(3).$

. . . .

$b_n = P(n) - b_0 - b_1F_1(n) - \ldots - b_{n-1}F_{n-1}(n).$

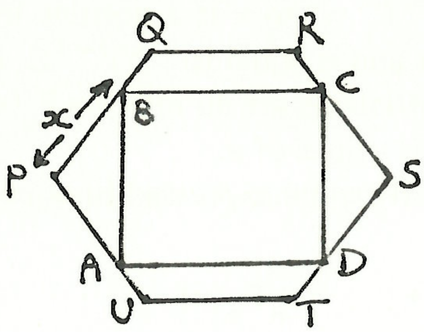
It is now clear that if P(x) is an integer for $x = 1, 2, 3, \ldots n$ then $b_0, b_1, b_2, \ldots b_n$ are all integers. QED.

O200 The hole in the head of a spanner is a regular hexagon with sides of length a. The spanner is used to tighten a square nut with sides of length b. Find conditions satisfied by a and b for this to be possible.

Answer One condition is that the diagonal of the square ($\sqrt{2}b$) must exceed the distance between opposite sides of the hexagon ($\sqrt{3}a$), since otherwise the nut can turn freely inside the spanner.

$$\sqrt{2}b > \sqrt{3}a \qquad (i)$$

Another condition is that the nut must not exceed in size the largest square

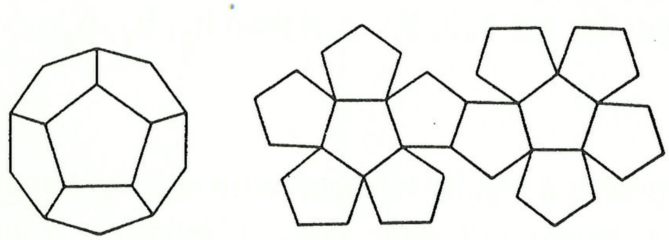


which can be inscribed in a hexagon of side a. This square (which has sides parallel to two sides of the hexagon) is shown in the figure. Using elementary trigonometry one easily sees using ΔABP that $b = \sqrt{3}x$ and using BCRQ that $b = 2(\frac{a-x}{2}) + a$.

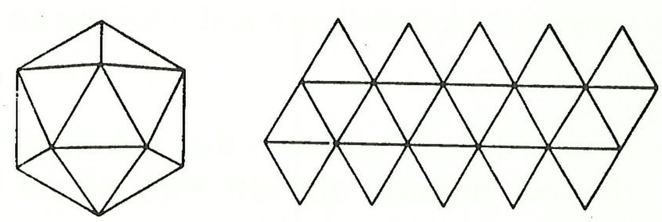
Eliminating x from these gives $b = (3 - \sqrt{3})a$ for this largest square. Hence $(\sqrt{3}/\sqrt{2})a < b \leqslant (3-\sqrt{3})a$ are the required conditions.

Solvers of Problems 191–200 (Vol 8 No 3)

H. Burns (Campbell High, ACT) 191–3.
R. Casley (Gosford High) 194 (good), 195, 196 (excellent), 198.
J. Holten (East Hills Boys' High) 192, good attempt at 193.
R. Sharp (Newington College) 200 (almost right).
C. Wood (Newington College) 194, 195, 200.
K. Burns (Campbell High, ACT) 194–200.
H. de Feraudy (St. Joseph's College, Hunters Hill) 194–199.
R. Kuhn (Sydney Grammar) 194, 195, 198, 200.



Dodecahedron {5, 3}



Icosahedron {3, 5}

Answer: Think 'n Drink: 250 miles.