SOLUTIONS OF PROBLEMS Q782 - Q792

Q.782 Let $S_n = \frac{1}{1^2-\frac{1}{4}} + \frac{1}{2^2-\frac{1}{4}} + \frac{1}{3^2-\frac{1}{4}} + \cdots + \frac{1}{n^2-\frac{1}{4}}$

Simplify this expression, and show that when n is large S_n is approximately equal to 2.

ANSWER

$$\frac{1}{k^2-\frac{1}{4}} = \frac{1}{(k-\frac{1}{2})(k+\frac{1}{2})} = \frac{1}{(k-\frac{1}{2})} - \frac{1}{(k+\frac{1}{2})}$$

$$\therefore S_n = \left(\frac{1}{1-\frac{1}{2}} - \frac{1}{1+\frac{1}{2}}\right) + \left(\frac{1}{2-\frac{1}{2}} - \frac{1}{2+\frac{1}{2}}\right) + \left(\frac{1}{3-\frac{1}{2}} - \frac{1}{3+\frac{1}{2}}\right)$$
$$+ \left(\frac{1}{(n-1\frac{1}{2})} - \frac{1}{n-\frac{1}{2}}\right) + \left(\frac{1}{n-\frac{1}{2}} - \frac{1}{n+\frac{1}{2}}\right)$$
$$= \frac{1}{1-\frac{1}{2}} - \frac{1}{n+\frac{1}{2}} = 2 - \frac{2}{2n+1}$$

When n is large, $\frac{2}{2n+1}$ is small, and S_n is close to 2.

Q.783 In the following equation:-

$$908 + 47 + 26 + 13 + 5 = 999$$

the left hand side contains every digit exactly once. Either find a similar expression (involving only + signs) whose sum is 1000, or prove that it is impossible to do so.

ANSWER The L.H.S. of the expression in the example can be written as

$$9 \cdot 10^2 + 0 \cdot 10^1 + 8 \cdot 10^0 + 4 \cdot 10^1 + 7 \cdot 10^0 + 2 \cdot 10^1 + 6 \cdot 10^0 + 1 \cdot 10^1 + 3 \cdot 10^0 + 5 \cdot 10^0.$$

Since $a \cdot 10^n = a \cdot (10^n - 1) + a = 9q + a$

(where $q = a \times \overbrace{111\cdots 1}^{n \text{ digits}}$, or $q = 0$ if $n = 0$), one can deduce that

$a \cdot 10^n + b \cdot 10^m + c \cdot 10^k + \cdots$ exceeds a multiple of 9 by $a + b + c + \cdots$

Hence any such expression in which $a, b, c \cdots$ are an arrangement of $\{0, 1, 2, \cdots, 9\}$ exceeds a multiple of 9 by $0 + 1 + 2 + \cdots + 9 = 45$, so is a different multiple of 9.

As 1000 is not a multiple of 9, there is no such expression equal to 1000.

Q.784 In the equation $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ all the coefficients $a_0, a_1, \cdots, a_n$ are odd integers. If the equation has a rational root (i.e. there are integers p, q such that $x = \frac{p}{q}$ satisfies the equation) prove that n also must be odd.

ANSWER Since any common factor of p and q may be cancelled out, we may assume that p and q are not both even. Replacing x in the equation by $\frac{p}{q}$ and multiplying through by q^n yields $a_0q^n + a_1pq^{n-1} + a_2p^2q^{n-2} + \cdots + a_np^n = 0$.

If p is even, q odd all terms on the L.H.S. are even whole numbers except the first, which is odd. Hence this sum, being odd, cannot be zero. This situation therefore is impossible, and an identical argument also eliminates the possibility p odd, q even.

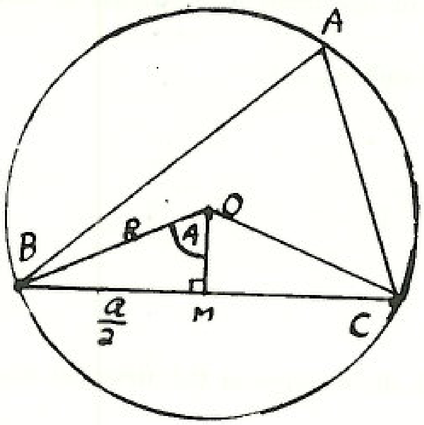
The only remaining possibility is the case when both p and q are odd. Then every term on the LHS is an odd whole number, and for an even sum the number of terms, $n + 1$, must be even.

Thus n must be odd.

Q.785 The circumcircle of ΔABC has radius not greater than 1cm. Show that the sum of the areas of squares with sides AB, AC and BC does not exceed $9cm^2$.

ANSWER The sine rule for the triangle asserts that $\frac{a}{\sin \widehat{A}} = \frac{b}{\sin \widehat{B}} = \frac{c}{\sin \widehat{C}} = 2R$ where R is the radius of the circumcircle.

Fig 1



(In Fig. 1, it is clear from ΔOBM that $\frac{a}{2} = R\sin\hat{A}$. The proof for obtuse angled triangles is similar).

Hence $AB^2 + BC^2 + CA^2 = c^2 + a^2 + b^2 = 4R^2(\sin^2\hat{A} + \sin^2\hat{B} + \sin^2\hat{C}) \leq 4(\sin^2\hat{A} + \sin^2\hat{B} + \sin^2\hat{C})$ when $R \leq 1$.

So it suffices to show that for any triangle $\sin^2\hat{A} + \sin^2\hat{B} + \sin^2\hat{C} = \frac{9}{4}$. We assume without loss of generality that $\hat{C}$ is acute.

$$\begin{aligned}
\sin^2 A + \sin^2 B + \sin^2 C &= \frac{1-\cos 2\hat{A}}{2} + \frac{1-\cos 2\hat{B}}{2} + \sin^2\hat{C} \\
&= 1 - \frac{1}{2}(\cos 2\hat{A} + \cos 2\hat{B}) + \sin^2\hat{C} \\
&= 1 - \cos(\hat{A}+\hat{B})\cos(\hat{A}-\hat{B}) + \sin^2\hat{C} \\
&= 1 + \cos\hat{C}\cos(\hat{A}-\hat{B}) + \sin^2\hat{C} \quad (\text{since } \hat{A}+\hat{B} = 180 - \hat{C}) \\
&\leq 1 + \cos\hat{C} + (1-\cos^2\hat{C}) \quad (\text{since } \cos(\hat{A}-\hat{B}) \leq 1 \,\&\, \cos\hat{C} > 0) \\
&\leq 2 + \frac{1}{4} - (\cos\hat{C} - \frac{1}{2})^2 \\
&\leq \frac{9}{4}.
\end{aligned}$$

Q.786 $\{a_1, a_2, a_3, \cdots, a_n, \cdots\}$ is an increasing list of positive integers. A second list $\{b_1, b_2, \cdots b_n, \cdots\}$ is defined by the rule:- b_n is the number of members of the first list which are less than n.

Show that any positive integer N may be expressed either as $N = a_n + n$ for some n or else as $N = b_n + n$

(For example if $\{a_n\} = \{2, 4, 6, 8, \cdots, 2n, \cdots\}$ then $\{b_n\} = \{0, 0, 1, 1, 2, 2, 3, 3, \cdots\}$. If $N = 9$, then it is expressible as $a_3 + 3$ but not as $b_n + n$ for any n.

If $N = 10$ it is expressible as $b_7 + 7$, but not in the form $a_n + n$.)

ANSWER Suppose N is any integer not expressible in the form $a_n + n$. To be more explicit, suppose it lies between $a_{k-1} + (k-1)$ and $a_k + k$.

$$\text{i.e. } a_{k-1} + (k-1) < N \leq (a_k + k) - 1 \qquad (1)$$

Let t be an integer such that $a_{k-1} < t \leq a_k$. (2)

Then $b_t = k - 1$ (since $a_1, a_2, \cdots, a_{k-1}$ are all the entries in the first list which are less than t).

$$\therefore b_t + t = (k-1) + t$$

Taking $t = N - k + 1$ (note that because of (1), this t satisfies the condition (2))

$$b_t + t = (k-1) + N - k + 1 = N.$$

Thus any integer not expressible in the form $a_n + n$ is expressible as $b_t + t$.

On the other hand, if $N = a_n + n$ you can check that

$b_{a_n} + a_n = (n-1) + a_n = N - 1$; whilst $b_{a_n+1} + (a_n + 1) = n + (a_n + 1) = N + 1$.

Thus there is no t such that $b_t + t = N$.

Q.787 Let $a_1, a_2, \cdots a_n$ be any positive numbers.

If $P = (i_1, i_2, \cdots i_n)$ is any arrangement of $\{1, 2, 3, \cdots n\}$ let

$$x_P = \frac{1}{a_{i_1}(a_{i_1} + a_{i_2})(a_{i_1} + a_{i_2} + a_{i_3}) \cdots (a_{i_1} + a_{i_2} + \cdots + a_{i_n})}$$

Show that the sum of all $n!$ of the numbers x_P is equal to $\frac{1}{a_1 a_2 \cdots a_n}$

(e.g. if $n = 2$ $\frac{1}{a_1(a_1+a_2)} + \frac{1}{a_2(a_2+a_1)} = \frac{1}{a_1 a_2}$. This is easy to check. You are asked to prove the corresponding result for any positive integer n.)

ANSWER Proof by mathematical induction on n.

Induction Hypothesis: We assume that the stated result is true when $n = k$, (k is any given whole number).

Induction Step: Consider any set of $(k+1)$ positive numbers $S = \{a_1, a_2, \cdots a_{k+1}\}$ and form the sum $\sigma = \sum x_p$ of all $(k+1)!$ terms of the type given. All these terms have the common factor $\frac{1}{a_1+a_2+a_3+\cdots+a_{k+1}}$.

$\therefore \sigma = \frac{1}{(a_1+a_2+\cdots+a_{k+1}})F$ where for each value of $t = 1, 2, 3, \cdots, k+1$ there are $k!$ terms in the factor F which do not contain the number a_t. By our induction hypothesis the sum of these terms of F is equal to $\frac{1}{\text{product of } S-\{a_t\}} = \frac{a_t}{a_1 a_2 \cdots a_t \cdots a_{k+1}}$.

$$\text{Hence } F = \sum_{t=1}^{k+1} \frac{a_t}{a_1 a_2 \cdots a_{k+1}} = \frac{a_1 + a_2 + \cdots + a_{k+1}}{a_1 a_2 \cdots a_{k+1}}$$

$$\therefore \sigma = \frac{1}{(a_1 + a_2 + \cdots + a_{k+1})} \frac{(a_1 + a_2 \cdots + a_{k+1})}{(a_1 a_2 \cdots a_{k+1})}$$

$$= \frac{1}{a_1 a_2 \cdots a_{k+1}}.$$

We have now shown that if the stated result is true for $n = k$, it is true for $n = k+1$.

Since it is trivially true for $n = 1$, it is true for all positive integer values of n.

Q.788 Six dollar coins are arranged in a triangle (see Fig.1). Show how they may be brought into the circular formation in Fig.2. An allowable move consists in sliding one coin, without disturbing the others, to a new position in which it touches two others.

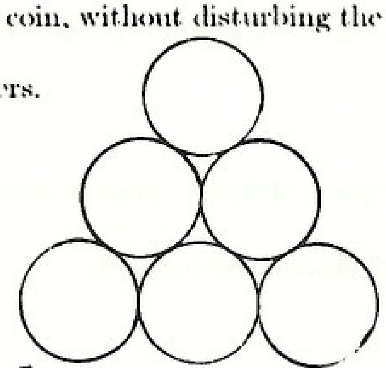


Fig 1

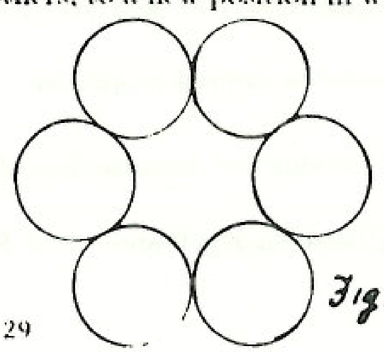
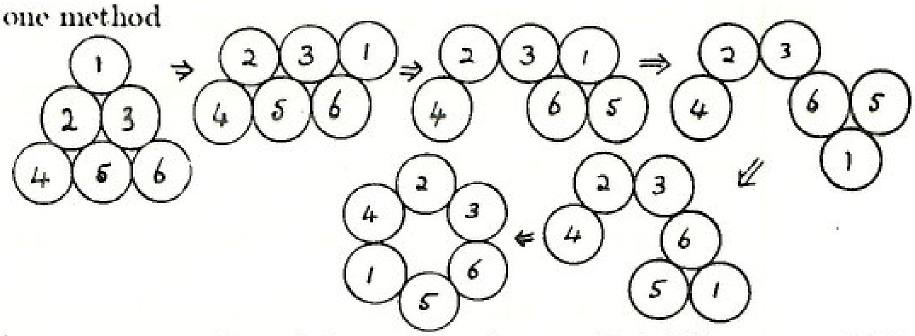


Fig 2

ANSWER Here is one method



Q.789 The sum of seven consecutive whole numbers is a perfect fifth power. If the smallest and the largest are deleted the sum of the remaining five is a perfect cube. Can you find such a set of integers?

ANSWER Let x be the middle number. From the data we deduce immediately that $7x$ is a perfect fifth power and $5x$ is a perfect cube.

Let $x = 5^k 7^\ell p^\alpha q^\beta \cdots$ be the factorisation of x into prime factors.

($p, q \cdots$ are primes other than 5 & 7).

Since $7x$ is a perfect fifth power, $k, \ell + 1, \alpha, \beta \cdots$ are all multiples of 5.

Since $5x$ is a perfect cube $k + 1, \ell, \alpha, \beta \cdots$ are all multiples of 3.

$\therefore \alpha, \beta, \cdots$ are multiples of 15.

Let $k = 5t$. Then $5t + 1$ is a multiple of 3 $\Leftrightarrow 6t - (t - 1)$ is a multiple of 3

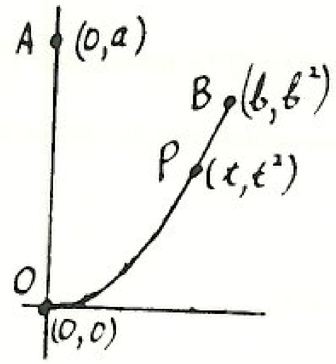
$\Leftrightarrow t = 3s + 1,\ k = 5 + 15s$ for some whole number s.

Similarly $\ell = 9 + 15u$ for some whole number u.

The smallest value of x is obtained by taking, $s, u, \alpha, \beta, \cdots$ equal to 0.

This gives $x = 5^5 7^9$; and $\{x - 3, x - 2, x - 1, x, x + 1, x + 2, x + 3\}$ is the smallest set with the desired properties.

Q.790 Find the shortest distance from the point $A(0, a)$ to the arc of the parabola $y = x^2$ lying between $x = 0$ and $x = b$, for all numbers a and b.



ANSWER For $0 \le t \le b$ the point $P(t, t^2)$ lies on the parabola $y = x^2$.

Let $f(t) = AP^2 = (0 - t)^2 + (a - t^2)^2$

$$= t^4 - (2a - 1)t^2 + a^2.$$

We wish to find the minimum of $f(t)$ for $0 \le t \le b$.

$f'(t) = 4t^3 - 2(2a - 1)t$.

This vanishes for $t = 0$, and, when $a > \frac{1}{2}$, also for $t = \sqrt{a - \frac{1}{2}}$. By examining the sign of $f'(t)$ near $t = 0$ (or, alternatively the sign of $f''(0)$) we find that $f(t)$ has a minimum at $t = 0$ if $a \le \frac{1}{2}$, but a local maximum at $t = 0$ if $a > \frac{1}{2}$.

Thus when $a \le \frac{1}{2}$ the shortest distance from A to the parabola is a, the distance from $(0, a)$ to $(0, 0)$.

When $a > \frac{1}{2}$, the stationary point of $f(t)$ at $t_0 = \sqrt{a - \frac{1}{2}}$ is a minimum since $f'(t) = 4t(t - t_0)(t + t_0)$ which changes from negative to positive as t increases through t_0. Thus if $a > \frac{1}{2}$ and if $b \ge t_0$, the point $P(t_0, t_0^2)$ is the closest point to $A(0, a)$ and $AP = \sqrt{t_0^2 + (t_0^2 - a)^2} = \sqrt{a - \frac{1}{4}}$.

Finally if $b < \sqrt{a - \frac{1}{2}}$, $f'(t) < 0$ for all t in $[0, b]$, and the end point $B(b, b^2)$ is closest to A. Then $AB = \sqrt{b^4 - (2a - 1)b^2 + a^2}$ is the shortest distance.

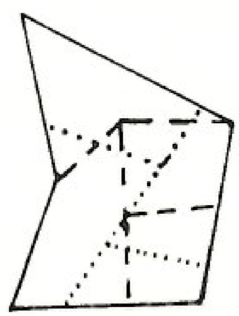
Q.791 (i) Given any polygon, of area Acm^2, prove that it can be chopped up into a finite number of smaller polygons which can be reassembled, jigsaw fashion, to make a rectangle of dimension $1cm \times Acm$.

(ii) Given any two polygons both of area Acm^2, prove that one can be dissected into a finite set of smaller polygons which can be reassembled to make a figure congruent to the other.

ANSWER If P and Q are two polygons we shall write $P \equiv Q$ to mean that P can be chopped into a finite number of smaller polygons which can be reassembled to obtain a figure congruent to Q.

Th(S) It is at once obvious that if $P \equiv Q$ then also $Q \equiv P$.

Th(T) It is not difficult to see that if R is a third polygon such that $P \equiv Q$ and $Q \equiv R$ then also $P \equiv R$.



(Proof:- In the figure the broken lines - - - indicate the dissection of Q implied by the relation $P \equiv Q$, and the dotted lines .... the dissection of Q required for $Q \equiv R$. If cuts are made along both sets of lines a finite number of small polygons is obtained which can be reassembled to give either P or R. Hence $P \equiv R$).

It is now obvious from Th(T) that (ii) follows from (i), by taking for P and R the two given polygons of area Acm^2 and for Q a rectangle of dimensions $1cm \times Acm$. It remains to prove (i). Note that repeated use of Th(T) yields

Corollary (T). If $P_1, P_2, \cdots, P_n$ are polygons such that $P_1 \equiv P_2$, $P_2 \equiv P_3, \cdots$, and $P_{n-1} \equiv P_n$, then $P_1 \equiv P_n$.

It is obvious enough that any polygon can be dissected into triangles. (In fact if the polygon has n vertices it can be dissected into $(n-2)$ triangles. This is immediate for convex polygons, and we have proved it in an earlier issue of Parabola for polygons with re-entrant angles).

Suppose P, a polygon of area Acm^2, is dissected into k triangles of areas $A_1, A_2, \cdots A_k$ where $A_1 + A_2 + \cdots + A_k = A$. If we show that for each $i = 1, 2, \cdots k$ a triangle of area $A_i \equiv a$ rectangle of dimensions $1cm \times A_i cm$, then (i) is established since the k rectangles can be stacked to make a rectangle of size $1cm \times Acm$.

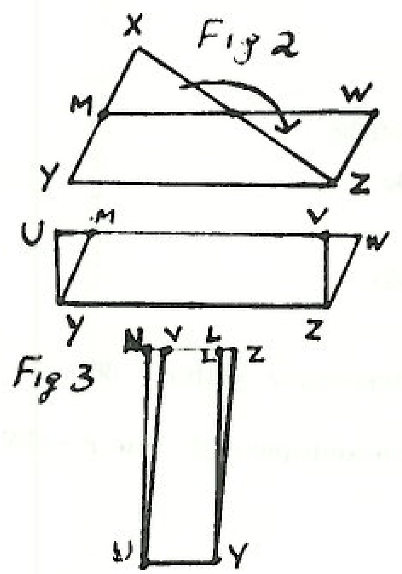


Fig. 2 indicates how a triangle, ΔXYZ, can be reassembled into a parallelogram $YZW'M'$ and Fig. 3 shows how that parallellogram can be reassembled into a different parallellogram on the same base. If necessary this can be repeated, until the parallellogram is either a rectangle, or arbitarily close to a rectangle, with the length YU rational, say $\frac{h}{k}$cms. The same process with YU as base now enables us to obtain a rectangle, $YULM$.

If this rectangle is sliced by vertical cuts into h slices, each a rectangle on a base of length $\frac{1}{k}$ cm, these can be stacked to give a single rectangle on a base of that length. Finally we chop this into k equal pieces by horizontal cuts, and stand them side by side to obtain a rectangle on a base of 1cm.

In view of Corollary (T), we have succeeded in showing that a triangle can be reassembled into a rectangle on a base of 1cm, and our proof of (i) is complete.

Q.792 There are altogether 120 students at Diophantos High School, the boys outnumbering the girls. To replace breakages in the chemical laboratory, following an unfortunate incident involving a football, each of the boys contributed 30 cents, and each girl 2 cents. The amount collected exactly covered the cost of 12 test tubes at 7 cents each and a number of beakers at 90 cents each.

How many beakers?

ANSWER Let g, b, x be the numbers of girls, boys, and beakers respectively. We are given

$$b + g = 120$$

$$30b + 2g = 7 \times 12 + 90x.$$

Eliminating b, $30(120 - g) + 2g = 84 + 90x$.

$$1758 = 14g + 45x \tag{1}$$

We require a solution of (1) in positive integers g, x, with $g < 60$.

From (1) x must be even, and g must be a multiple of 3. Put $x = 2X$, $g = 3G$ and divide through by 6.

$$293 = 7G + 15X. \tag{2}$$

Since $0 < G < 20$, $\frac{293-7\times 20}{15} < X < \frac{293}{15}$

$$\text{so } 10 < X < 20. \tag{3}$$

From (2) $X = 7(41 - G - 2X) + 6$, so X leaves the remainder 6 on division by 7. In view of (3), we must have $X = 13$, so that the number of beakers, x, is $2 \times 13 = 26$.

(Also from (2), $G = 14$ so the number of girls is $3G = 42$, and the number of boys is $120 - 42 = 78$).