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POLYNOMIAL APPROXIMATIONS

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Many mathematical functions are not readily computable using only a finite sequence of elementary arithmetic operations (additions, subtractions, multiplications and divisions). In this article, I will discuss a general strategy for evaluating a function $f(x)$ for x in a chosen interval I. This strategy is the basis for many algorithms used by electronic calculators and computers.

The idea is very simple: given f and I, we look for a polynomial of degree n, denoted by P_n, such that

$$f(x) \approx P_n(x) \quad \text{for } x \in I. \tag{1}$$

(The symbol $\approx$ means "approximately equal to.") Since the evaluation of

$$P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

requires only a finite number of multiplications and additions (or subtractions), (1) provides a computable approximation to $f(x)$.

Most methods for choosing $P_n(x)$ have the property that the approximation (1) improves as one increases the degree n. Of course, increasing the degree involves doing more work, both in finding the coefficients a_0, a_1, ..., a_n, and in evaluating $P_n(x)$ for a particular x. The trick is to achieve the required accuracy using the smallest possible n.

As an example, consider the natural logarithm of $1 + x$. It turns out that

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} \quad \text{for } x \in I, \tag{2}$$

where I is a "sufficiently small" interval about 0. If we were interested in, say, $\ln 1{\cdot}1037$, then we would put $x = 0{\cdot}1037$ and obtain

$$\ln 1{\cdot}1037 \approx 0{\cdot}1037 - \frac{(0{\cdot}1037)^2}{2} + \frac{(0{\cdot}1037)^3}{3} \doteq 0{\cdot}09869. \tag{3}$$

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(The dot over the equal sign indicates that we have rounded to the number of decimal places shown.) According to my calculator, $\ln 1{\cdot}1037 \doteq 0{\cdot}09867$, so our approximation (3) is probably accurate to 4 decimal places. For an explanation of why (2) holds, let

$$f(x) = \ln(1+x) \quad \text{and} \quad P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3},$$

and calculate the first few derivatives,

$$\begin{aligned} f'(x) &= (1+x)^{-1}, & P_3'(x) &= 1 - x + x^2, \\ f''(x) &= -(1+x)^{-2}, & P_3''(x) &= -1 + 2x, \\ f'''(x) &= 2(1+x)^{-3}, & P_3'''(x) &= 2. \end{aligned}$$

When $x = 0$, we have

$$f(0) = 0 = P_3(0), \quad f'(0) = 1 = P_3'(0), \quad f''(0) = -1 = P_3''(0), \quad f'''(0) = 2 = P_3'''(0),$$

so the values of f and its derivatives of order 1, 2 and 3 all agree with those of P_3, forcing the graphs of the two functions to be close together in an interval around 0; see the computer-generated plots in Figure 1.

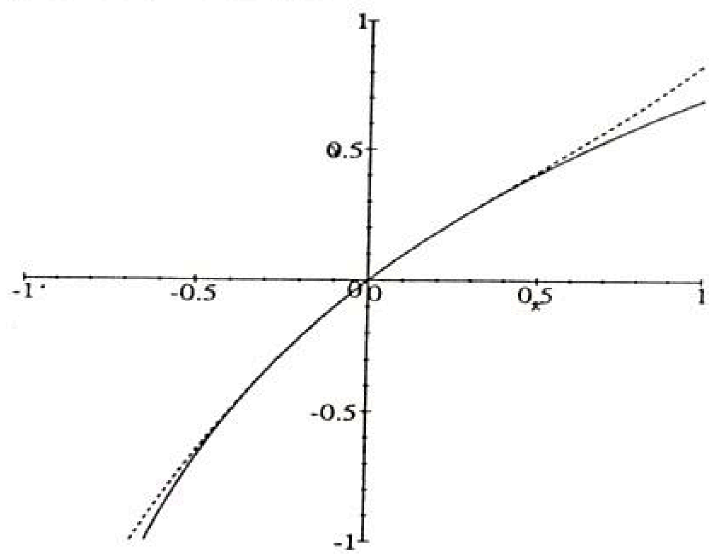

Figure 1: The solid line is $\ln(1+x)$, the dashed line is $x - x^2/2 + x^3/3$

In general, you can easily check that if we put

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + f^{(n)}(0) \leq vern!x^n, \tag{4}$$

where $n! = 1 \times 2 \times 3 \times \cdots \times n$, then

$$P_n(0) = f(0), \quad P_n'(0) = f'(0), \quad \ldots, \quad P_n^{(n)}(0) = f^{(n)}(0).$$

We call (4) the *Taylor polynomial* of degree n for the function $f(x)$. In the example above, $x - x^2/2 + x^3/3$ is the Taylor polynomial of degree 3 for $\ln(1+x)$. As an exercise, you might like to verify that

$$\begin{aligned} e^x &\approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}, \\ \sin x &\approx x - \frac{x^3}{3!} + \frac{x^5}{5!}, \\ \cos x &\approx 1 - \frac{x^2}{2} + \frac{x^4}{4!}. \end{aligned}$$

The Taylor approximation $f(x) \approx P_n(x)$ deteriorates as x moves away from 0, as you can see from Figure 1. If we are interested in values of $f(x)$ for x near a point c at some distance from 0, then we can work with $f(c+x)$ instead of $f(x)$, because $c+x$ is close to c when x is close to 0. The resulting approximation is

$$f(c+x) \approx f(c) + f'(c)x + \frac{f''(c)}{2}x^2 + \cdots + \frac{f^{(n)}(c)}{n!}x^n;$$

obviously, to use this Taylor polynomial we must know the values of $f(x)$ and its first n derivatives at $x = c$.

Any particular Taylor polynomial can only give accurate approximations for x in a small interval around a single point. If we want to cover a larger interval, then we can use several Taylor polynomials, or perhaps a single Taylor polynomial of very high degree. However, neither option is very convenient, and in practice it is better to use other types of polynomials.

For instance, consider the following cubic approximation to $\ln(1+x)$,

$$P_3(x) = 0{\cdot}0004416160 + 0{\cdot}9834928213\,x - 0{\cdot}4000352894\,x^2 + 0{\cdot}1096896488\,x^3. \tag{5}$$

If you were to plot this polynomial together with $\ln(1+x)$ for $0 \le x \le 1$, then the two graphs would be indistinguishable. To see any difference at all, we have to plot the error, $P_3(x) - \ln(1+x)$, and magnify the vertical scale; see Figure 2. The cubic (5) is an example of what is called a *minimax* polynomial: out of all possible approximations of the form $P_3(x) = a_0 + a_1x + x_2x^2 + x_3x^3$, this particular choice minimises the maximum error over the interval $[0, 1]$, i.e., the coefficients in (5) minimise the quantity

$$\max_{0 \le x \le 1} |(a_0 + a_1x + a_2x^2 + a_3x^3) - \ln(1+x)|. \tag{6}$$

From Figure 2, you can see that this minimax error is less than 0·00045.

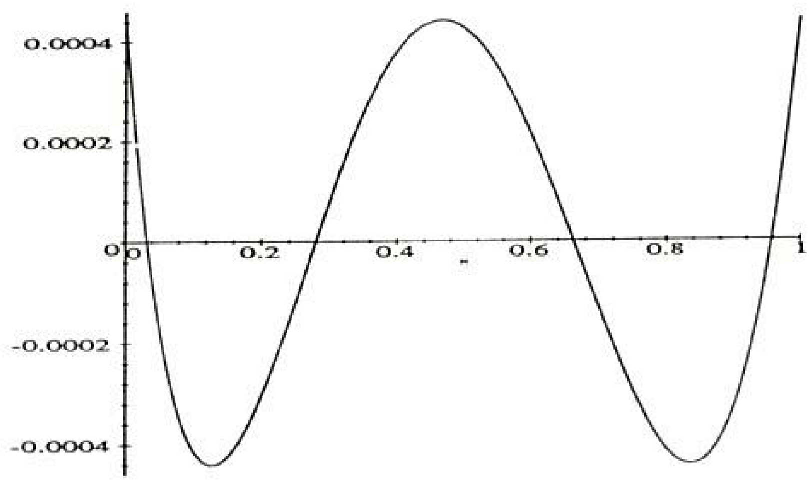

Figure 2: The error $P_3(x) - \ln(1+x)$ for the minimax polynomial (5)

The minimax polynomial can be found using a method known as the *Remez algorithm*. The details are rather complicated, but the general idea is to start with some reasonable first attempt at a polynomial approximation — e.g., a Taylor polynomial — and then keep modifying the coefficients until the error looks like Figure 2, where all the local maximima and minima are of the same magnitude. It turns out that this *equi-oscillation property* guarantees that the maximum error is as small as possible. You can think of the Remez algorithm as spreading the error uniformly over the given interval, in contrast to the error for the Taylor polynomial which is highly non-uniform; see Figure 3, where the maximum error over the interval $[0, 1]$ is about 0·14, or more than 300 times larger than for the minimax polynomial (5).

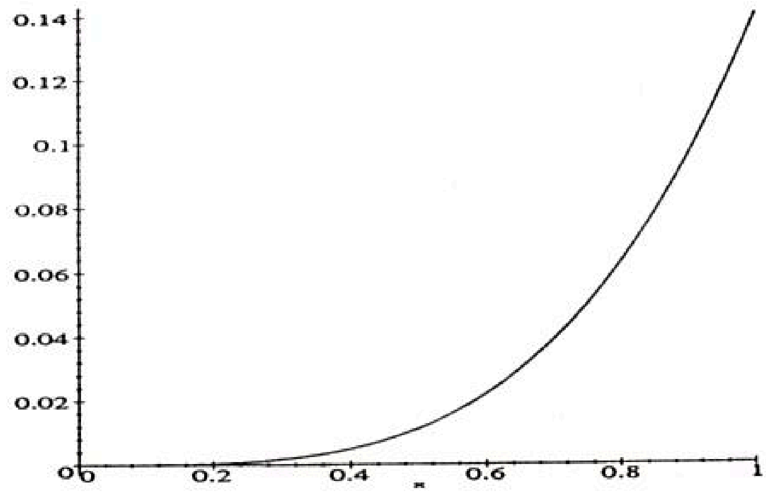

Figure 3: The error $P_3(x) - \ln(1+x)$ for the Taylor polynomial $x - x^2/2 + x^3/3$.

I should point out that in order apply the Remez algorithm, it is necessary to have a primary method of computing $f(x)$ that uses only elementary arithmetic operations. One might use Taylor polynomials of high degree, or sometimes there are clever, special-purpose techniques for particular functions. For example, in a previous article (**Parabola** 28(2)) I described a method for computing $\ln x$ and e^x, based on the identities

$$\ln(xy) = \ln x + \ln y \quad \text{and} \quad e^{x+y} = e^x e^y.$$

Once the coefficients of the minimax polynomial $P_n(x)$ are known, the primary method for computing $f(x)$ is no longer necessary — for a given x, evaluating $P_n(x)$ will almost certainly involve less work.

There is one final matter I want to discuss. Many functions take particularly simple values at special points; for instance,

$$\ln 1 = 0, \quad e^0 = 1, \quad \sin 0 = 0, \quad \cos 0 = 1.$$

It is usually desirable that polynomial approximations should reproduce such relations exactly. Thus, a drawback of the minimax polynomial (5) is that it gives 0·00044... as its value for ln 1. To get around this problem, we can minimise the maximum error (6) subject to the constraint $a_0 = 0$, so that $P_3(0) = 0$ exactly. The resulting cubic is

$$P_3(x) = 0{\cdot}9874532942\,x - 0{\cdot}4084070196\,x^2 + 0{\cdot}1146352802\,x^3, \qquad (7)$$

and a plot of its error is shown in Figure 4.

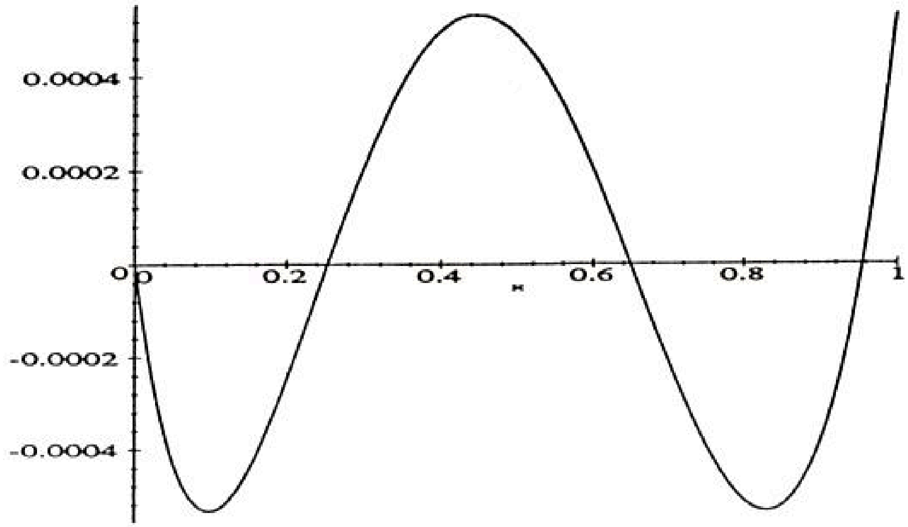

Figure 4: The error $P_3(x) - \ln(1 + x)$ for the constrained minimax polynomial (7).

The maximum error is about 20% larger than for the unconstrained minimax cubic (5), but for most purposes this overall loss of accuracy is a price worth paying in order to obtain the exact value for $\ln 1$. In any case, the error can always be reduced by increasing the degree while still requiring that $a_0 = 0$.

SOLUTION TO PUZZLE

The passage is Fermat's famous marginal note in Diophantus' **Arithmetic** (as transcribed in a later edition of Diophantus published by Fermat's son):

"However a cube cannot be divided into [the sum of] two cubes, nor a fourth power into two fourth powers, nor in general any power greater than a square into two equal powers: of which fact I have found a marvellous proof. But the space in the margin is too narrow [to write the proof down]."

Now see page 24.