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PROBLEM SECTION

You are invited to submit solutions to any or all of the following problems, accompanied by your name, school and year or form. Solutions of these problems will appear in the next issue of **Parabola**; your solution(s) may be used if they are received in time.

Q.941 On my desk calendar two numbers are given: the number of days in the year up to today, and the number remaining. (For example, on New Year's Day the numbers were 1 and 364.) What date could it be if the two numbers have the same digits, possibly in a different order? (No number may start with a zero.) Will the answer to this question be the same next year?

Q.942

1. Find positive integers x, y such that

$$x^2 - 2y^2 = -1.$$

2. Show that if (x, y) is a solution of the above equation then $(3x + 4y,\ 2x + 3y)$ is also a solution.

3. Prove that there are infinitely many non-negative integers n such that $n^2+(n+1)^2$ is a square.

Q.943 Prove that if $a_2, a_2, \cdots, a_n$ and $x_1, x_2, \cdots, x_n$ are positive numbers than

$$(a_1x_1 + a_2x_2 + \cdots + a_nx_n)(\frac{a_1}{x_1} + \frac{a_2}{x_2} + \cdots + \frac{a_n}{x_n}) \geq (a_1 + a_2 + \cdots + a_n)^2.$$

Under what conditions does equality hold?

Q.944 If n is large, find a simple approximate formula for

$$\sqrt{1 - \frac{1}{n^2}} + \sqrt{1 - \frac{4}{n^2}} + \sqrt{1 - \frac{9}{n^2}} + \cdots + \sqrt{1 - \frac{(n-1)^2}{n^2}} + \sqrt{1 - \frac{n^2}{n^2}}$$

Q.945 A regular polygon with n sides is inscribed in a circle. If A, B, C and D are four successive vertices of the polygon then the length of AD equals the side of the polygon plus the radius of the circle. Find all possible values of n.

Q.946

1. Show that if n is an integer, $n \geq 0$, then $2^{4n+2} + 1$ is divisible by 5.

2. Factorise $x^{4n} + 4$ into a product of two polynomials.

3. Show that if $n \geq 2$ then $\dfrac{2^{4n+2}+1}{5}$ is composite.

Q.947 Three series of equidistant parallel lines are drawn in a plane, each serie forming an angle of $60°$ with the other two; the plane is thus covered with a network of equilateral triangles. Is it possible to find four of the intersection points of these lines which form the vertices of a square?

Q.948 An infinite sequence of real numbers $a_1, a_2, a_3, \cdots$ is defined by choosing some value of a_1 and specifying

$$a_{n+1} = \frac{a_n + c}{1 - ca_n}$$

for $n \geq 1$, where c is constant. Prove that for every integer $k > 2$, a constant c can be found such that the sequence is periodic and has period k. (A sequence is called periodic if at some stage it repeats itself from the beginning; the period of the sequence is the smallest possible number of steps before the repetition begins. For example the sequence $5, 7, 1, 2, 5, 7, 1, 2, 5, 7, 1, 2, \cdots$ has period 4.)