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PROBLEM SECTION

You are invited to submit solutions to any or all of the following problems, accompanied by your name, school and year or form. Solutions of these problems will appear in the next issue of **Parabola**; your solution(s) may be used if they are received in time.

Hint. Some of the following problems are similar to, or based on, problems from this year's School Mathematics Competition. Solutions to the competition problems will be found elsewhere in this issue of **Parabola**.

Q. 949 (a) We have a collection of numbers, each of which is either zero or one. Not all of the numbers are the same, and the total number of elements in the collection is prime. It is permitted to choose any two or more of the numbers (but not the whole collection) and replace each of them by the average of the chosen numbers. Show that no matter how often we perform this replacement operation we shall never reach a situation in which all numbers in the collection are the same.

(b) For this question recall that the *geometric mean* of two positive numbers x and y is defined to be $\sqrt{xy}$.

A collection of 1995 numbers consists of 1994 twos and a one. It is permitted to choose any two numbers from the collection and replace each of them by the geometric mean of the two. Is it possible by repeating this operation to obtain a collection in which all 1995 numbers are the same?

Q. 950 Show that any triangle of one of the following types can be dissected into three isosceles triangles:

(a) acute-angled triangles;

(b) triangles with at least one 45° angle;

(c) triangles with one angle five times another;

(d) triangles with one angle six times another;

(e) triangles with one angle seven times another.

Q. 951 Let $a_1, a_2, a_3, \dots$ be a sequence of positive integers with the property that $a_1 \leq a_2 \leq a_3 \leq \cdots$. Define a second sequence $b_1, b_2, b_3, \dots$, where b_k is the number of terms among $a_1, a_2, a_3, \ldots$ which are less than k. Prove that for any positive integer m, if n denotes the value a_m then

$$(a_1 + a_2 + \cdots + a_m) + (b_1 + b_2 + \cdots + b_n) = mn .$$

Q. 952 Each of two ellipses passes through the two foci of the other. Prove that

(a) the four foci lie at the vertices of a parallelogram;

(b) if the focal lengths of the two ellipses are equal, then the ellipses are congruent.

Q. 953 Let a and b be unequal rational numbers. Show that

(a) if a and b are positive and $\sqrt{a} - \sqrt{b}$ is rational, then $\sqrt{a}$ and $\sqrt{b}$ are rational;

(b) if $\sqrt[3]{a} - \sqrt[3]{b}$ is rational, then $\sqrt[3]{a}$ and $\sqrt[3]{b}$ are rational.

Q. 954 A cyclist sets off from a point 0 and rides with constant velocity v along a straight highway. A messenger, who is at a distance a from point 0 and at a distance b from the highway wants to deliver a letter to the cyclist. What is the minimum velocity with which the messenger should run in order to achieve this outcome assuming she starts running at the same time the cyclist leaves 0?

Q. 955 Prove that the polynomial $x^{44} + x^{33} + x^{22} + x^{11} + 1$ is divisible by the polynomial $x^4 + x^3 + x^2 + x + 1$.

Q. 956 A party of four hikers who walk at 6kph and one motor cyclist who travels at 30kph leave town A simultaneously on a journey to town B, which is 45kms from A.
The motor cyclist can carry one passenger and carries each hiker a part of the journey and then returns for the others in turn. Find the minimum time required for the whole party to reach town B, and find how far each pedestrian has to walk.

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Problem Solvers

Belinda Gotley, Year 10, All Saints Anglican School, (Merrimac, Qld), solved Question 942.

Peter Kanka, Year 12, James Cook Boys' Technology High School, solved Questions 941, 943 and 945.