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UNDERSTANDING MATHEMATICAL INDUCTION

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Introduction

This article has two aims:

(i) To clear up some misunderstanding about mathematical induction;

(ii) To draw attention to the fact that mathematical induction is an axiom and to explain the significance of this.

First, here is a simple example of a proof by mathematical induction. The proof is for the following statement (S).

(S): $1+2+3+\ldots+n=\frac{1}{2}n(n+1)$, for all positive integers n.

STEP 1

$$\text{When } n=1, \qquad LHS = 1$$
$$RHS = \frac{1}{2}.1.(1+1) = 1$$

Since $LHS = RHS$, (S) is true for $n=1$.

STEP 2 Suppose that (S) is true for $n=k$ $\quad (k \in J^+)$.

Thus,

$$1+2+3+\ldots+k=\frac{1}{2}\ k(k+1)\ldots\ldots\ldots(A)$$

Then prove (S) is true for $n=k+1$, i.e., prove

$$1+2+3+\ldots+k+(k+1)=\frac{1}{2}(k+1)(k+2).$$

$$\begin{aligned} LHS &= 1+2+3+\ldots+k+(k+1) \\ &= \frac{1}{2}k(k+1)+(k+1) \quad [\text{from}(A)] \\ &= \frac{1}{2}(k+1)(k+2) \\ &= RHS \end{aligned}$$

Conclusion Since (S) is true for $n=1$, and true for $n=k+1$ if true for $n=k$, (S) is true for all $n \in J^+$.

Mathematical induction is not a case of inductive reasoning

It is important to understand that mathematical induction is not induction, just like a pink elephant is not an elephant. Mathematical induction has nothing to do with inductive reasoning. In a widely used senior school textbook mathematical induction is introduced by way of a brief explanation of inductive reasoning, which is described as deriving a general statement from one or more particular cases.

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In fact the basic difference between induction and deduction has little to do with arguing from particular to general or vice versa. The basic difference is that deduction is demonstrative whereas induction is not. Consider the following examples:

DEDUCTION :

1. If the litmus paper turns red, the liquid is an acid.
2. The litmus paper is turning red.

∴ The liquid is an acid.

To say that this argument is demonstrative means that it is impossible for the premisses 1. and 2. to be true and the conclusion to be false. We say that the argument is deductively valid.

INDUCTION:

1. Zebra_1, has stripes.
2. Zebra_2 has stripes.
3. Zebra_3 has stripes.

$\vdots$

1000. Zebra_{1000} has stripes.

∴ All zebras have stripes.

This is an example of inductive reasoning. Even if we accept the argument as a sound piece of inductive reasoning, the reasoning is not demonstrative. For in this case it is possible for all the premisses $1, 2, 3, \ldots, 1000$ to be true and the conclusion to be false. (The 1001-st Zebra might not have stripes.)

Mathematical induction is actually deductive. In our earlier example, if (S) true for $n = 1$, and true for $n = k + 1$ if true for $n = k$, then (S) must be true for all $n \in J^+$. Mathematical induction is demonstrative reasoning. Reference to inductive reasoning in this context only leads to confusion and is best omitted.

Mathematical induction as an axiom.

Mathematical induction is sometimes regarded as a method or process. We often say "Use the method of mathematical induction to prove...." which is not objectionable provided it is clearly understood that in fact mathematical induction is an axiom. It is the fifth of Peano's five axioms for the "natural numbers".[2] Here are Peano's axioms:

(1) 1 is a positive integer.

(2) If n is a positive integer, then there is another positive integer n' called the successor of n.

(3) $1 \neq n'$ for any positive integer n.

(4) If $n' = m'$ then $n = m$.

(5) Let G be a set of positive integers such that

[2] Peano's axioms were for the cardinals and included zero. I use the positive integers here because most inductions start at $n = 1$.

(i) 1 belongs to G,

(ii) n' belongs to G whenever n belongs to G.

Then G is the set of positive integers.

Peano's five axioms define the concept "positive integer" -mathematical induction is one of those axioms.

Any difficulty in understanding mathematical induction usually arises out of STEP 2. Thus, it is sometimes asked: even if (S) is true for $n = k + 1$ when true for $n = k$, why must it be true for all positive integers? The best way of answering this question is by reference to Peano's axioms. Peano's axioms are an attempt to describe in a minimal way the essential properties of the positive integers. Consider axioms (1)–(4) first. These four axioms describe a set S which contains 1, and in which every element has a unique successor. The set of positive integers has this property if we take the successor n' of any number n to be $n + 1$, but so does, for example, the set of positive real numbers. Any set with these properties is necessarily infinite, since otherwise some element would not have a successor, or would share its successor with some other element in contravention of (4). Axiom (5) says that J^+, the set of positive integers, is the smallest set with these properties. This last property ensures that J^+ is uniquely defined by Peano's axioms.

Let us now consider why mathematical induction works. Given a proposition, such as our earlier (S), we let G be the set of positive integers which it is true. (G is called the truth set of the proposition.) In proving (S) by induction we establish it is true when $n = 1$, which means $1 \in G$, and that, if it is true when $n = k$ then it is also true when $n = k + 1$, which means that $k' \in G$ whenever $k \in G$. Consequently according to axiom (5) G must be J^+.

In conclusion we can say: (1) Mathematical induction is induction by name only. It has little to do with inductive reasoning. (2) Mathematical induction is not a somewhat arbitrary, though convenient, method introduced to prove certain general statements, such as (S), about the positive integers. Mathematical induction is an axiom which is part of the very definition of the concept of a positive integer.