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MATHEMATICS: USEFUL TOOL, JUST GOOD CLEAN FUN ... OR BOTH?

Peter Coutis[1]

In high school we learn some interesting mathematics and develop some (potentially) very useful skills. But how exactly are these skills and techniques applied to the real world? Every day in science and industry, people with strong mathematical backgrounds develop systems of equations (known as **mathematical models**) which are used to predict the future development of the system being studied. Such models are presently employed to describe phenomena as diverse as large-scale ocean circulation and tropical cyclone path prediction, to the design of super-sonic aircraft and the prediction of stock market fluctuations! Although the models vary in complexity and in design from one application to the next, the underlying principle of sound mathematical analysis is the common thread that ties them together.

In practice, the closer the mathematical model is made to reality, the more difficult it becomes to obtain solutions. In fact, for most real world applications, obtaining analytic (mathematical) solutions is impossible! This is where the computer comes in. With the onset of the computer age, mathematical techniques to find approximate, but accurate solutions to large, complicated systems of equations have developed tremendously. This branch of mathematics is loosely referred to as 'numerical methods' and is designed to make use of the incredible computational power of today's supercomputers.

Mathematical models used to describe physical systems often require functions which depend on space and time. The equations that make up the models typically involve functions of one or more variables and their derivatives. They are known collectively as **differential equations**. You are probably already familiar with some simple differential equations. For example,

$$\frac{dy}{dt} = ky \tag{1}$$

is the law of exponential growth, $(k > 0)$ or decay $(k < 0)$, and

$$\frac{d^2x}{dt^2} = -n^2x \tag{2}$$

[1]Peter Coutis has recently moved from the School of Mathematics, UNSW to the School of Mathematical Sciences, UTS

is the (differential) equation governing simple harmonic motion. The importance of differential equations was not lost on Marius Sophus Lie, one of the nineteenth century's greatest mathematicians, who said: "Among all the mathematical disciplines the theory of differential equations is the most important. It furnishes the explanation of all those elementary manifestations of nature which involve time."

Let us now look at an example of a mathematical model. Since our main purpose here is to illustrate the methods involved we will keep mathematical complexity down to a bare minimum. Before we can set up our model we do, of course, need a problem to solve. So here it is:

You are placed in charge of a team instructed to design a single-person vehicle that is as fuel efficient as possible. Knowing that both friction and weight will reduce fuel economy, you design a vehicle that is light compared to the combined weight of the driver and the fuel, and you minimise friction by using as few moving parts as possible. You also make the vehicle as aerodynamic as you can. You are told that to test the fuel economy, you are required to record the amount of fuel used by the vehicle for a $20km$ journey. Unfortunately for you, however, the fuel gauge malfunctions and the only information you gather from the test run is that the vehicle can go $40km$ on a full tank of fuel. Now, can we set up a model to predict (as accurately as possible) how much fuel was used for the first $20km$?

To get started, we need to make some assumptions based on our intuitive feel for the situation. It is realistic to assume that the *rate* of fuel consumption is directly proportional to the total vehicle weight. We can express the instantaneous fuel consumption rate therefore, as $\dfrac{dL}{dX}$ litres per km, where X is the distance travelled in km and L is the fuel consumed in litres. The weight of the vehicle (including fuel) at any time is given by $W + D(L_0 - L)$, where we will define W as the (predetermined) combined weight of the driver and vehicle, and D as the fuel density. L_0 is the initial fuel volume. Note that when $L = 0$, i.e. at the beginning of the test run, the total weight is simply the combined weight $W + DL_0$ of the driver and vehicle plus the weight of the fuel. Our original assumption relating fuel consumption to instantaneous vehicle weight may be expressed mathematically as:

$$\frac{dL}{dX} = B[W + D(L_0 - L)] \tag{3}$$

Where, B is an (as yet) undetermined constant of proportionality. Ultimately, our aim is to obtain L as a function of X. Now, clearly $L_0 \geq L$ for the entire journey and the amount of fuel consumed at the start and end are (respectively)

$$L(0) = 0 \tag{4}$$

$$L(40) = L_0. \tag{5}$$

Our solution will come from the correct use of equations (3), (4) and (5). So let's get started.

The first thing we notice about equation (3) is that it is possible to rewrite it with X's and constants on one side and L's and constants on the other. We do this by dividing by $[W + D(L_0 - L)]$ and multiplying by dX to produce

$$BdX = \frac{dL}{W + D(L_0 - L)}. \tag{6}$$

To proceed further, we simply integrate both sides of (6), yielding (after a little algebra which you should try)

$$W + D(L_0 - L) = Ce^{-DBX}. \tag{7}$$

Here, C is a new (but unknown) constant. At this stage, we have two unknown constants (B and C), but fortunately we also have equations (4) and (5) up our sleeve. Hence, the procedure for finding the unknown constants will just reduce to solving two equations in two unknowns. Let us first apply equation (4), which is known as an **initial condition**. By substituting $X = 0$ and $L = 0$ into equation (7), we get

$$C = W + DL_0 \tag{8}$$

Now put (8) into (7) and you should come up with

$$L(X) = G - Ge^{-DBX} \tag{9}$$

where $G = \dfrac{W}{D} + L_0$. But what about B? For this we need to apply equation (5), which says that $L = L_0$ when $X = 40$. Substitute this into (9) and see if you can show that:

$$L_0 = G - Ge^{-40DB}$$
$$40DB = ln(DG/W) = ln(1 + \frac{DL_0}{W}) \tag{10}$$

We can now use (10) to find the value of our original constant of proportionality, B. Note that DL_0 is the original fuel weight and recall that W is the total combined weight of the vehicle and driver. Hence, the value of the proportionality constant is intimately linked to the ratio of the fuel weight to the vehicle weight (as you would expect!).

We are now in a position to answer the question posed at the beginning of the problem. How much fuel did the vehicle use for the first $20km$? The answer is easily obtained by substituting $X = 20$ into (9). It is useful to try some numbers in (9) to get a quantitative feel for the solution. Let's assume that $W = 100kg,\ L_0 = 20L$ and $D = 1kg/L$. Then, from (10):

$$B = \frac{1}{40} ln\left(1 + \frac{20}{100}\right) = 0.00456. \tag{11}$$

So,

$$L(20) = G - Ge^{-20DB}. \tag{12}$$

With $G = \left(\dfrac{100}{1} + 20\right) = 120$ and $B = 0.00456$, (12) tells us that $L(20) = 10.46L$. This means that the vehicle uses slightly more than half of its fuel in the first half of the journey. (Why?). It is also enlightening to examine the properties of the solution given by (9). The graph of $L(X)$, (from (9)), appears in Figure 1. Note that $L = 0$ at $X = 0$ and that L is an increasing function of X. (Of course, X is also an increasing function of L, since the larger the amount of fuel expended, the greater the distance travelled). A horizontal asymptote appears at $L = G$, but this is physically unreasonable since we only have L_0 litres of fuel to begin with. (You can check that when $L = G$, the weight of the fuel is $-W$, i.e. negative the total vehicle weight, giving the vehicle a weight of zero!). While you may guess that a vehicle of zero mass *may* travel forever, as is suggested by the asymptote, mathematically and *physically*, we reject this case as impossible.

It is interesting to note that $L(X)$ has very similar properties to a function known as the logistic function. The logistic function is the solution of a differential equation (similar in nature to equation (3)) commonly used to model population growth and was first used for this purpose by the Dutch mathematical biologist Verhulst in the 1840's. In terms of population dynamics, the logistic function suggests that population growth cannot go on unchecked, but instead is limited by death rates, food supply, etc. In population dynamics, this is how we interpret the horizontal asymptote in Figure 1.

As a check on our solution to the fuel consumption problem, we also examine the properties of equation (10). In general, if we let the maximum range of the vehicle on L_0 litres of fuel be X_m, then (10) can be written as

$$X_m = \frac{1}{DB} ln(1 + \frac{DL_0}{W}) \tag{11'}$$

Note in particular, that X_m *increases* as L_0 increases and that X_m *decreases* as W increases. Also, $X_m = 0$ for $L_0 = 0$, i.e. the vehicle goes nowhere if you don't put fuel in it! These properties are intuitively correct which should give us confidence in our solution.

OK, so what exactly have we achieved here? Well, aside from solving the problem using a sound mathematical and intuitive approach, we have shown that with some basic knowledge of calculus and a little common sense, understanding a range of interesting physical problems is quite within our reach. Perhaps then, you can be convinced that the answer to the question posed in the title is BOTH!

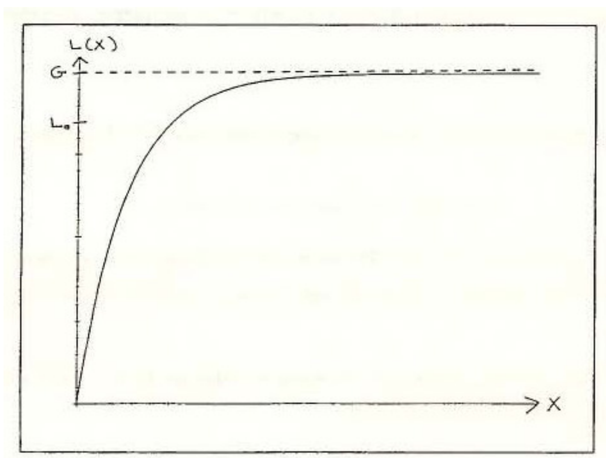

Figure 1