Parabola Volume 32, Issue 3 (1996)

A NOTE ON CONIC SECTIONS

Frank Reid[1]

David Rowe, a Year 12 student at Barker College, rang us recently with the following question. We thought that it would be of interest to our readers.

> We have a right circular cone of semi-vertical angle α, and it is well known that any section by a plane produces a conic. David asked, *Would the axis of the cone pass through a focus of the curve?*

We can answer this question in the following way.

Consider a circular cone of semi-vertical angle α and vertex V. This cone is cut by a plane Σ, where Σ makes angle θ with the vertical. The plane Σ meets the surface of the cone in a curve. Let P_1 be any point on this curve.

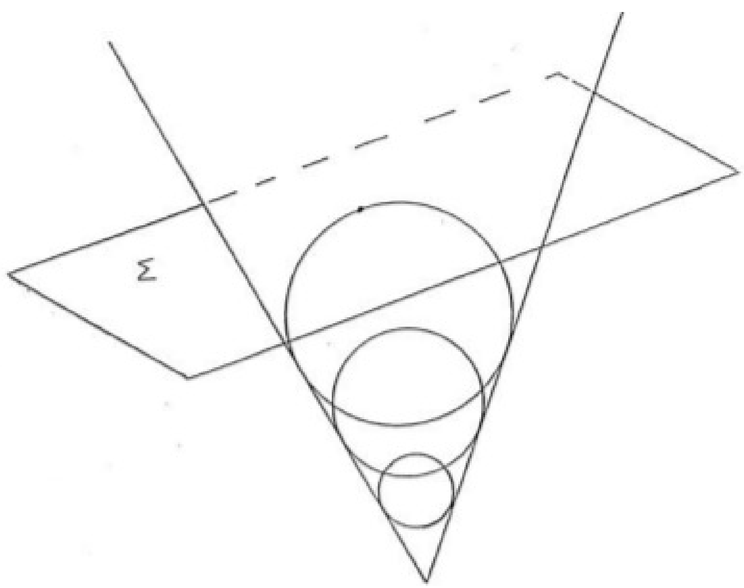

Figure 1

Imagine the cone turned upside down and a small spherical balloon resting in the cone. We can imagine that the balloon is now blown up bigger and bigger and so it will rise up the cone, always touching the surface of the cone. We can continue blowing up the spherical balloon until it just touches the plane Σ, and also touches the sides of the cone as in figure 1.

This means that, given a circular cone and a plane Σ, there exists a sphere which touches the plane Σ and the cone.

[1]Frank Reid is a school teacher from St. Ursula's College, Kingsgrove, who is spending a year at the School of Mathematics at the University of N.S.W.

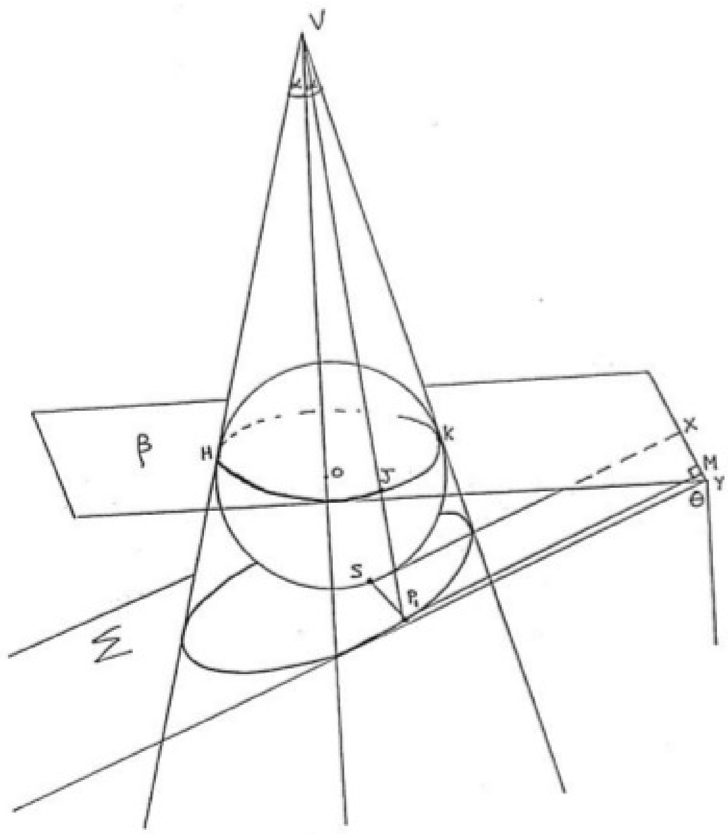

Figure 2

In figure 2 this sphere is $HJKS$ with centre O and circle of contact HJK in a plane β, where β meets the plane Σ in a line XY. S is the point of contact of the sphere with the plane Σ.

If P_1 is the intersection of the line VJ with the plane Σ and P_1M is the perpendicular from P_1 to XY, then it is possible to express the distance of P_1 from the plane β in two different ways:

(i) As the projection of P_1J on OV: since P_1J lies along the cone, it makes angle α with OV and so its projection onto OV is $P_1J\cos\alpha$.

(ii) As the projection of P_1M on OV : if θ is the angle between P_1M and the vertical, then θ is the angle between P_1M and OV and so the projection onto OV is $P_1M\cos\theta$.

Now these are two expressions for the same distance from P_1 to β and so

$$\begin{aligned} P_1J\cos\alpha &= P_1M\cos\theta \\ P_1J &= \frac{P_1M\cos\theta}{\cos\alpha}. \end{aligned}$$

But $SP_1 = P_1J$ since each is a tangent to the sphere. (Tangents to a sphere from an external point are equal). So

$$SP_1 = P_1J = \frac{P_1M\cos\theta}{\cos\alpha} = eP_1M \text{ where } e = \frac{\cos\theta}{\cos\alpha}$$

and this relation is independent of the position chosen for P_1 on the curve of section.

Therefore, the curve on which P_1 lies is a conic with focus S and directrix XY, and the axis of the cone does not pass through the focus. In fact *the focus is the point of contact of the plane Σ and the sphere.*

Similarly, there exists a sphere on the other side of the plane which touches the sides of the cone, and touches the plane at a point S'. We leave it to the reader as an exercise to show, in a very similar way, that P_2 lies on a conic, with focus S'. (Let β' be a plane through $H'K'$, parallel to β and let $X'M'$ be the line in which Σ cuts the plane β'. Express the distance of P_2 to the plane β' in two different ways.)

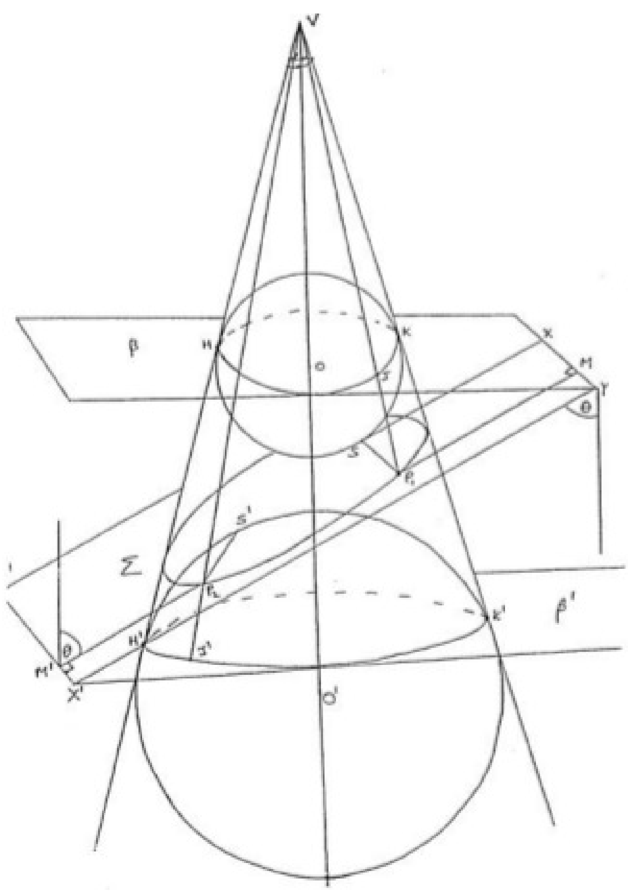

Figure 3

Thus S and S' are the foci of the conic, but the axis of the cone does not pass through either.

If $\theta = \alpha$, then $e = 1$, and the plane Σ is parallel to a generator of the cone; i.e. the curve is a parabola.

If $\theta > \alpha$, then $\cos\theta < \cos\alpha$ and $e < 1$; i.e. the curve in this case is an ellipse.

If $\theta < \alpha$, then $\cos\theta > \cos\alpha$, and $e > 1$; i.e. the curve is a hyperbola.

Of course, if $\theta = 90^\circ$ then Σ is perpendicular to the axis of the cone and cuts the cone in a circle. The axis of the cone passes through the centre of this circle.

Thanks, David for an interesting question. If you have a question you would like answered, write to us and we will see what we can do [Ed].