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# AREA OF A POLYGON

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Our aim is to find a simple formula for the area of a polygon whose vertices are

$$A_1(x_1, y_1), \quad A_2(x_2, y_2), \quad \cdots \quad A_n(x_n, y_n),$$

joined in that order.

**1. The Area of a Triangle:**

Initially we concentrate on finding the area of a triangle with vertices $O(0,0)$, $A(x_1, y_1)$, $B(x_2, y_2)$ as shown.

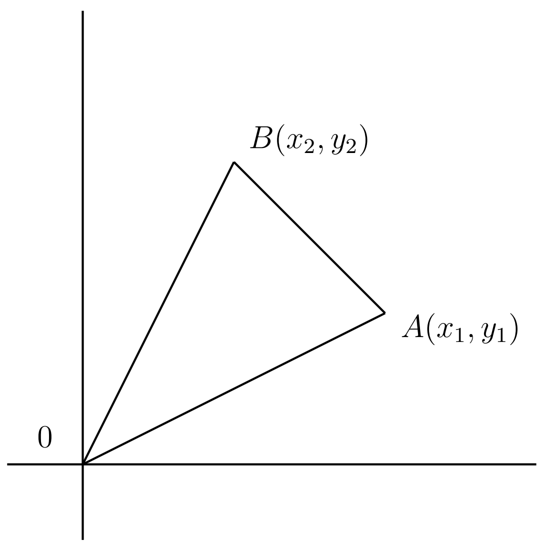



**Figure 1**

It is not difficult to show using elementary co-ordinate geometry that the area of the triangle is given by

$$\text{Area} \;= \frac{1}{2}|x_1 y_2 - x_2 y_1|.$$

[1] Mr. Carlos Alberto da Silva Victor is one of our correspondents from Rio de Janeiro in Brazil. He used the formula given in this article to solve problem 967 in **Parabola** Vol.32, No.1, and sent us this article to explain it. At his request, his English has been tidied up.

(If you know about vectors, you can show this result using the cross product of the vectors $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$, since the cross product gives the area of the parallelogram formed from these vertices, and so we halve it to find the area of the triangle.)

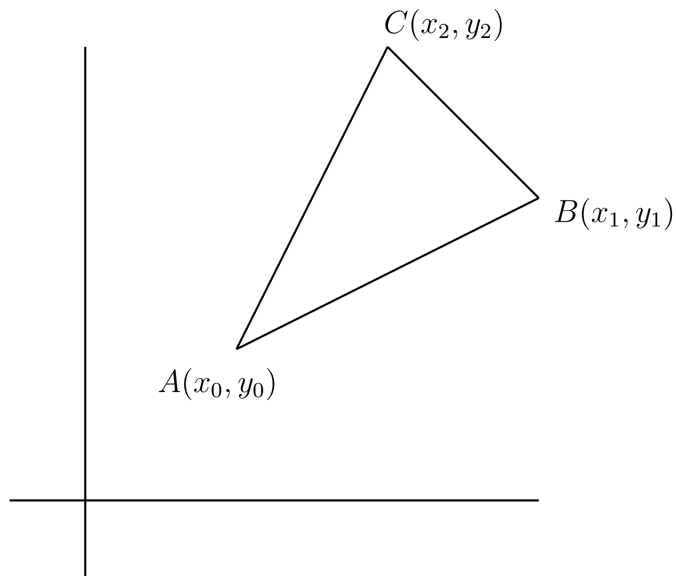



**Figure 2**

By moving the origin to the point $(x_0, y_0)$, we can write the area of the triangle with vertices at $A(x_0, y_0)$, $B(x_1, y_1), C(x_2, y_2)$ as

$$\frac{1}{2}|(x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)|$$

which simplifies to

$$\text{Area} \;= \frac{1}{2}|x_0y_1 + x_1y_2 + x_2y_0 - (x_1y_0 + x_2y_1 + x_0y_2)|.$$

Formidable though this may appear, we can write the formula inside the absolute value sign, which we will call $H$, using the following notation.

$$H = \left| \begin{array}{cccc} x_0 & x_1 & x_2 & x_0 \\ & \times \quad \times \quad \times & & \\ y_0 & y_1 & y_2 & y_0 \end{array} \right|$$

$$\swarrow \qquad\qquad\qquad \searrow$$

$$-\,x_1y_0 \qquad\qquad\qquad +\,x_2y_0$$

where we add the product of the entries along each diagonal sloping to the right and subtract the sum of the product of the entries along each diagonal sloping to the left.

This value of $H$ depends on the order of the points (which we chose in an anti-clockwise order.)

For example, if we swap the points $(x_1, y_1)$ and $(x_2, y_2)$ we get

$$H' = \begin{vmatrix} x_0 & x_2 & x_1 & x_0 \\ & \times \quad \times \quad \times & & \\ y_0 & y_2 & y_1 & y_0 \end{vmatrix} = -H$$

$$\swarrow$$
$$-x_2 y_0$$

**2. The Area of a Polygon.**

Suppose now we have a convex polygon with $n$ vertices joined in the order given, and a point $K(x_0, y_0)$ inside the polygon as shown.

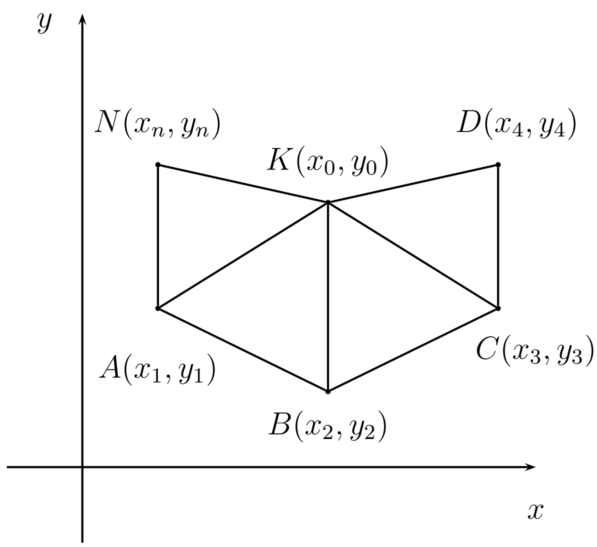



**Figure 3**

We divide the polygon into $n$ triangles, each with $K$ as one of its vertices, and we take the vertices in anti-clockwise direction. We can now apply the formula from section 1, to get the following expression for $2 \times$ Area :

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_0 \\ y_0 & y_1 & y_2 & y_0 \end{vmatrix} + \begin{vmatrix} x_0 & x_2 & x_3 & x_0 \\ y_0 & y_2 & y_3 & y_0 \end{vmatrix} + \begin{vmatrix} x_0 & x_3 & x_4 & x_0 \\ y_0 & y_3 & y_4 & y_0 \end{vmatrix} + \ldots + \begin{vmatrix} x_0 & x_n & x_1 & x_0 \\ y_0 & y_n & y_1 & y_0 \end{vmatrix}$$

We can expand this out to get

$$\begin{aligned} &\{(x_0y_1 + x_1y_2 + x_2y_0) - (x_1y_0 + x_2y_1 + x_0y_2)\} \\ &+\{(x_0y_2 + x_2y_3 + x_3y_0) - (x_2y_0 + x_3y_2 + x_0y_3)\} \\ &+\{x_0y_3 + x_3y_4 + x_4y_0) - (x_3y_0 + x_4y_3 + x_0y_4)\} \\ &+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+\{x_0y_n + x_ny_1 + x_1y_0) - (x_ny_0 + x_1y_n + x_0y_1)\} \end{aligned}$$

After cancelling terms, we get,

$$(x_1y_2 + x_2y_3 + ... + x_{n-1}y_n + x_ny_1) - (x_2y_1 + x_3y_2... + x_ny_{n-1} + x_1y_n)$$

and this can be rewritten back in the array notation as:

$$\text{Area} \;= \frac{1}{2}\left| \begin{array}{cccccc} x_1 & x_2 & x_3 & ... & x_n & x_1 \\ y_1 & y_2 & y_3 & ... & y_n & y_1 \end{array} \right|.$$

Observe that this is independent of the point $K$ we chose as the common vertex of the triangles.

One can similarly show, by a careful choice of triangles, that the above formula also works for non-convex polygons.

**Example 1:** Find the area of the polygon $ABCDE$ with vertices $A(1,1,), B(2,4), C(3,6), D(-1,8), E(-4,5)$.

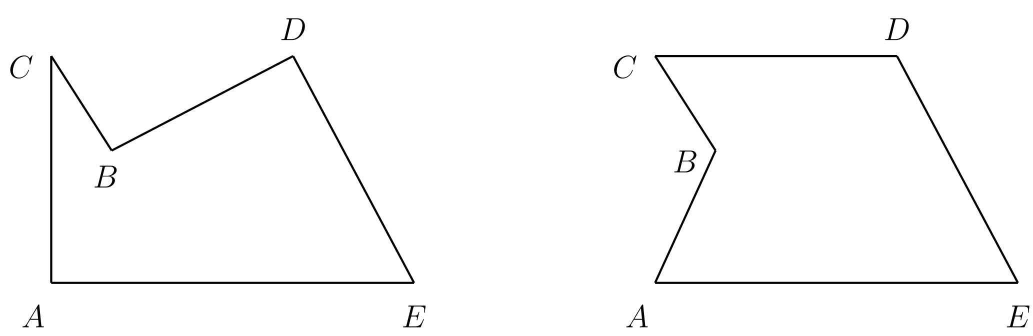
We compute

$$\frac{1}{2}\left| \begin{array}{cccccc} 1 & 2 & 3 & -1 & -4 & 1 \\ 1 & 4 & 6 & 8 & 5 & 1 \end{array} \right| = \frac{1}{2}\{(4+12+24-5-4)-(2+12-6-32+5)\} = 25 \text{ units}^2$$

**Example 2:** Suppose we take the points $A(0,0), B(4,2), C(0,8), D(6,12), E(10,0)$.

a. The area of the (non-convex) polygon $AEDBC$ is

$$\frac{1}{2}\left| \begin{array}{cccccc} 0 & 10 & 6 & 4 & 0 & 0 \\ 0 & 0 & 12 & 2 & 8 & 0 \end{array} \right| = 58$$




**Figures 4 & 5**

b. The area of the polygon $AEDCB$ is

$$\frac{1}{2}\left| \begin{array}{cccccc} 0 & 10 & 6 & 0 & -4 & 0 \\ 0 & 0 & 12 & 8 & 2 & 0 \end{array} \right| = 68.$$

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Think of a number between one and fifty. Double it, subtract sixty-one, add one, subtract the number you started with, close your eyes      Dark! isn't it?