Parabola Volume 32, Issue 3 (1996)

SOLUTIONS TO PROBLEMS 975-984

Q.975 For which real numbers x is it true that

$$[5x] = [3x] + 2[x] + 1\ ?$$

Here $[x]$ denotes the greatest integer less than or equal to x; for example, $[\pi] = 3$.

ANS. Let $x = a + y$, where a is an integer and $0 \leq y < 1$. Thus $a = [x]$ and the equation can be written

$$[5a + 5y] = [3a + 3y] + 2a + 1.$$

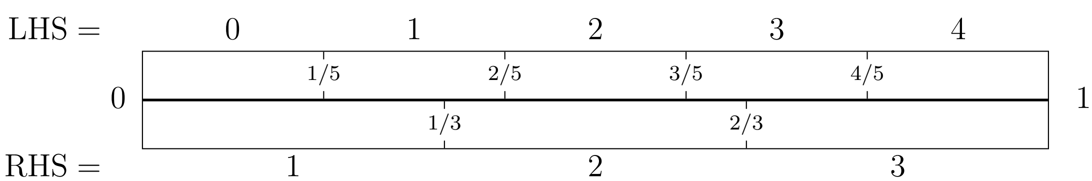
Now $[b + z] = b + [z]$ for any integer b, so we have

$$5a + [5y] = 3a + [3y] + 2a + 1$$

and so

$$[5y] = [3y] + 1.$$

We can visualise the LHS and RHS by drawing a number line for $0 \leq y \leq 1$.



The LHS and the RHS are equal if and only if

$$\frac{1}{5} \leq y < \frac{1}{3} \quad \text{or} \quad \frac{2}{5} \leq y < \frac{3}{5} \quad \text{or} \quad \frac{2}{3} \leq y < \frac{4}{5}.$$

Q.976 It was shown in problem 6 of the Senior Division that every power of 2 has a multiple whose decimal expansion contains only the digits 1 and 2. Find all pairs of non-zero digits which can replace 1 and 2 so that the statement is still true.

ANS. Let a, b be two different non-zero digits. Since any multiple of any power of 2 (other than 1) ends in an even digit, the required result cannot be true if a and b are both odd.

If one of the digits is odd and the other even then the result is true, and the proof is very similar to that given in the competition solutions. Suppose that a is even and b odd. For $k = 1$ we have

$$a = 2^k\,\frac{a}{2},$$

which is a one-digit multiple of 2^1 using only the digit a. If $n = 2^k m$ is a k-digit multiple of 2^k with only the digits a and b then

(i) if m is odd then

$$2^{k+1}\,\frac{5^k b+m}{2}=10^k b+n$$

is a $(k+1)$-digit multiple of 2^{k+1} with only a and b for digits;

(ii) if m is even then

$$2^{k+1}\,\frac{5^k a+m}{2}=10^k a+n$$

has the same property.

So, by induction, the result is true for all $k\geq 1$.

What happens if a,b are both even? Let 2^k be a power of 2. Then, from above, 2^k has a multiple n consisting of digits 2 and 3 only; hence $2n$ is a multiple of 2^k consisting of digits 4 and 6 only. Thus the pair $a,b=4,6$ will work. Similar arguments cover all pairs of even digits except $a,b=2,6$. Now a number containing twos and sixes only must end in the digits $22, 26, 62$ or 66 and therefore cannot be a multiple of 4, or of any higher power of 2.

To sum up: the result is still true for any pair of digits which are not both odd, except for 2 and 6.

Q.977 Evaluate

$$\frac{1\times 2^2}{2\times 3}+\frac{2\times 2^3}{3\times 4}+\frac{3\times 2^4}{4\times 5}+\frac{4\times 2^5}{5\times 6}+\cdots+\frac{n\,2^{n+1}}{(n+1)(n+2)}.$$

ANS. For any k we have

$$\frac{k}{(k+1)(k+2)}=\frac{2}{k+2}-\frac{1}{k+1}\,;$$

hence

$$\begin{aligned}
&\tfrac{1\times 2^2}{2\times 3}+\tfrac{2\times 2^3}{3\times 4}+\tfrac{3\times 2^4}{4\times 5}+\tfrac{4\times 2^5}{5\times 6}+\cdots+\tfrac{n2^{n+1}}{(n+1)(n+2)}\\
&\quad= 2^2\left(\frac{2}{3}-\frac{1}{2}\right)+2^3\left(\frac{2}{4}-\frac{1}{3}\right)+2^4\left(\frac{2}{5}-\frac{1}{4}\right)+2^5\left(\frac{2}{6}-\frac{1}{5}\right)+\cdots+2^{n+1}\left(\frac{2}{n+2}-\frac{1}{n+1}\right)\\
&\quad= -\frac{2^2}{2}+\frac{2^3}{3}-\frac{2^3}{3}+\frac{2^4}{4}+\frac{2^5}{5}-\frac{2^5}{5}+\frac{2^6}{6}+\cdots-\frac{2^{n+1}}{n+1}+\frac{2^{n+2}}{n+2}\\
&\quad= \frac{2^{n+2}}{n+2}-2.
\end{aligned}$$

Q.978 Show that three adjacent numbers in a row of Pascal's triangle can neither be in geometric progression nor in harmonic progression. (Three or more positive numbers are said to be in harmonic progression if their reciprocals are in arithmetic progression.)

ANS. Three consecutive numbers in a row of Pascal's triangle can be written

$$\binom{n}{r-1}, \binom{n}{r}, \binom{n}{r+1}$$

for some n, r with $n \geq 2$ and $1 \leq r \leq n-1$. These numbers are in geometric progression if and only if

$$\binom{n}{r}^2 = \binom{n}{r-1}\binom{n}{r+1},$$

that is,

$$\left(\frac{n!}{r!(n-r)!}\right)^2 = \frac{n!}{(r-1)!(n-r+1)!}\frac{n!}{(r+1)!(n-r-1)!}.$$

This simplifies to

$$(r+1)(n-r+1) = r(n-r)$$

or $n+1=0$, which is impossible.

The numbers are in harmonic progression if and only if

$$\frac{2}{\binom{n}{r}} = \frac{1}{\binom{n}{r-1}} + \frac{1}{\binom{n}{r+1}}.$$

This leads to

$$2\,\frac{r!(n-r)!}{n!} = \frac{(r-1)!(n-r+1)!}{n!} + \frac{(r+1)!(n-r-1)!}{n!}$$

and to

$$2r(n-r) = (n-r+1)(n-r) + (r+1)r$$

and hence, after some rearrangement, to

$$(n-2r)^2 + n = 0$$

which is impossible as $n > 0$.

Q.979 Let n be a positive integer. Find the remainder when $2^{3n} - 7n$ is divided by 49.

ANS. Using the Binomial Theorem,

$$\begin{aligned}
2^{3n} - 7n &= (7+1)^n - 7n \\
&= 7^n + \binom{n}{1}7^{n-1} + \cdots + \binom{n}{n-2}7^2 + \binom{n}{n-1}7 + 1 - 7n \\
&= 7^2\left(7^{n-2} + \binom{n}{1}7^{n-3} + \cdots + \binom{n}{n-2}\right) + 1,
\end{aligned}$$

and when this is divided by 49 the remainder is clearly 1.

Q.980 Charlie Chump, whose algebra is not very good, believes that (by cancelling the sixes)

$$\frac{16}{64} = \frac{1\not{6}}{\not{6}4} = \frac{1}{4}$$

Find all fractions involving two-digit integers which Charlie would correctly simplify (that is, which would be reduced to lowest terms by incorrectly cancelling a digit).

ANS. There are four cases to consider:

(a) $\frac{10n+a}{10n+b} = \frac{a}{b}$ (b) $\frac{10a+n}{10b+n} = \frac{a}{b}$ (c) $\frac{10n+a}{10b+n} = \frac{a}{b}$ (d) $\frac{10a+n}{1-n+b} = \frac{a}{b}$

where $n, a, b \in \{1,2,3,4,5,6,7,8,9\}$. We will only consider the cases where $a < b$.
(a) In the first, case,

$$\begin{aligned} 10nb + ab &= 10na + ab \\ a &= b \end{aligned}$$

and so the two numbers were the same.
(b) The second case is similar.
(c) In the third case,

$$\begin{aligned} 10nb + ab &= 10ab + na \\ 10nb &= 9ab + na < 9ab + nb \text{ (since } a < b) \\ 9nb &< 9ab \\ n &\le a - 1 \\ \text{so} \qquad na &= 10nb - 9ab \\ &\le 10(a-1)b - 9ab \\ &\le b(a-10) < 0 \end{aligned}$$

which is impossible.
(d) In the last case,

$$\begin{aligned} 10ab + nb &= 10na + ab \\ 9ab &= 10na - nb = (10a - b)n \end{aligned}$$

If 9 divides $10a - b = 9a + a - b$, then 9 divides $a - b$, which is impossible since $0 < a < b < 10$. Thus 3 must divide n and so $n = 3,\ 6$ or 9.

If $n = 3$, then $3ab = 10a - b$ and so $10a = 3ab + b = (3a+1)b$. Thus either b or $3a+1$ is divisible by 5, i.e. $b = 5,\ a = -1$ or $a = 3,\ b = 3$, neither of which is allowed.

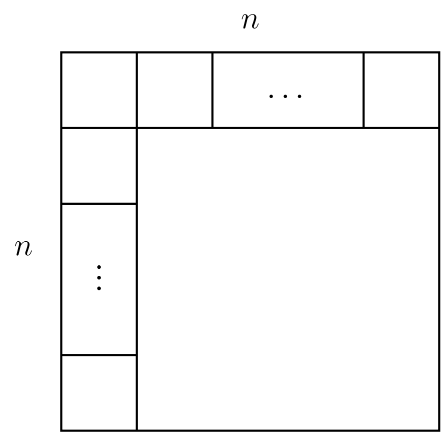
If $n = 6$, then $20a = 3ab + 2b = (3a+2)b$. Thus either b or $3a+2$ is divisible by 5, i.e. $b = 5,\ a = 2$ or $a = 1,\ b = 4$. If $n = 9$, then $10a = (a+1)b$ and so either b or $a+1$ is divisible by 5, i.e. $b = 5,\ a = 1$ or $a = 4,\ b = 8$ or $a = 0,\ b = 9$.

So the possible answers are

$$\frac{26}{65} = \frac{2}{5}, \quad \frac{16}{64} = \frac{1}{4}, \quad \frac{19}{95} = \frac{1}{5} \quad \text{and} \quad \frac{49}{98} = \frac{4}{8}.$$

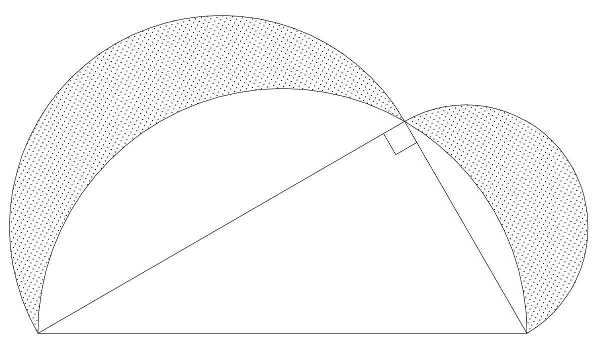
Q.981 Show that a square can be cut into n smaller squares (possibly of various sizes) for any $n > 5$.

ANS. To get $2(n+1)$ squares ($n \geq 1$), cut as follows:



To get $2n + 5 = 2(n+1) - 1 + 4$ squares, simply cut the top left-hand square into four.

Q.982 Semicircles are drawn internally on the hypotenuse of a right-angled triangle and externally on the other two sides.



Find, without using algebra or calculus, the total area of the shaded crescents.

ANS. The area of a semicircle is proportional to the area of the square drawn on its diameter. Hence, by Pythagoras' Theorem, the (area of the) semicircle on the hypotenuse equals the sum of the (areas of the) semicircles on the other two sides. From the diagram, the shaded region consists of the two smaller semicircular regions, plus the triangle itself, minus the semicircular region on the hypotenuse. Thus the total area of the shaded crescents equals the area of the triangle, that is, half the product of the two shorter sides.

Q.983 Show that there are integers a, b, c, not all zero, between -10^6 and 10^6, such that

$$-10^{-11} < a + b\sqrt{2} + c\sqrt{3} < 10^{-11}.$$

ANS. Each of the 10^{18} real numbers $r + s\sqrt{2} + t\sqrt{3}$, where $r, s, t \in \{0, 1, 2, \cdots 10^6 - 1\}$ are between 0 and $(1 + \sqrt{2} + \sqrt{3})10^6$. If we partition the interval $\{x : 0 \leq x < (1 + \sqrt{2} + \sqrt{3})10^6\}$ into $10^{18} - 1$ equal subintervals, then two of the above numbers $r_1 + s_1\sqrt{2} + t_1\sqrt{3}$ and $r_2 + s_2\sqrt{2} + t_2\sqrt{3}$ must fall in the same subinterval. If $a = r,\ -r_2,\ b = s_1 - s_2$ and $c = t_1 - t_2$, then

$$\begin{aligned} |a + b\sqrt{2} + c\sqrt{3}| &= |(r_1 + s_1\sqrt{2} + t_1\sqrt{3}) - (r_2 + s_2\sqrt{2} + t_2\sqrt{3})| \\ &< \frac{(1 + \sqrt{2} + \sqrt{3})10^6}{10^{18} - 1} \\ &< \frac{10^{17}}{10^{18}} = 10^{-11} \end{aligned}$$

Q.984 (Based on the article on page 12 of Vol. 32 No. 1). Let A, B be two sets of real numbers, each having cardinal number $\aleph_0$ and with no elements in common.

(a) Show that the cardinal number of the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

is $\aleph_0$.

(b) Show that the cardinal number of the set

$$\mathbb{R} - A = \{x : x \text{ is a real number and } x \notin A\}$$

is c.

(c) Show that the cardinal number of the set of all transcendental numbers is c.

ANS. (a) Let

$$\begin{array}{cccccc} 1, & 2, & 3, & \cdots, & n, & \cdots \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ a_1, & a_2, & a_3, & \cdots, & a_n, & \cdots \\ b_1, & b_2, & b_3, & \cdots, & b_n, & \cdots \end{array}$$

be $(1-1)$ correspondences between the positive integers and the elements of the two sets A and B.

Then

$$\begin{array}{cccccccccc} 1, & 2, & 3, & 4, & 5, & 6, & \cdots, & 2n-1, & 2n, & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \\ a_1, & b_1, & a_2, & b_2, & a_3, & b_3, & \cdots, & a_n, & b_n, & \cdots \end{array}$$

is a $(1-1)$ correspondence between the elements of $A \cup B$ and the positive integers.

(b) and (c)

Since the set of algebraic numbers (B) has cardinal number $\aleph_0$, and the set of all real numbers (A) has cardinal number c, (c) is a particular case of (b) and we will only prove (c). The proof can easily be modified to apply to the more general statement (b).

We have to show that it is possible to set up a $(1-1)$ correspondence between the set of transcendental numbers and the set of all real numbers.

Select from the transcendentals the set C (whose cardinal number is $\aleph_0$) consisting of the set $\{\pi,\ 2\pi,\ 3\pi,\ \cdots\ n\pi,\ \cdots\}$. The union of this set with the set of algebraic numbers B still has cardinal number $\aleph_0$ (by (a)), i.e. it is possible to set up a $(1-1)$ correspondence between the sets C and $B \cup C$. If such a $(1-1)$ correspondence is set up and extended by letting any transcendental number not in C correspond to itself, we have a $(1-1)$ correspondence between the transcendental numbers and the set of all real numbers.

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A CARD PREDICTION

The final card is the one on the bottom of the pack after they were shuffled. See if you can prove this (it only relies on the fact that there are 52 cards in a pack – so if some-one else wishes to try the trick, remove a card while handing them the pack).

HIGH-SPEED CALCULATION

The answer is just 11 times the fourth-last number (in the example given, $11 \times 55 = 605$). See if you can prove this.