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SOLUTIONS TO PROBLEMS 985-992

Q.985 For what values of the positive integer n is

$$\text{(a)}\quad 5n+2 \qquad\qquad\qquad \text{(b)}\quad 7n+2$$

a perfect square?

ANS. The following table lists the remainders $x^2 \pmod 5$ and $x^2 \pmod 7$ when the squares of the numbers $x = 0, 1, \cdots 6$ are divided by 5 and 7:

$x = 0$	1	2	3	4	5	6	7	8	$\cdots$
$x^2 = 0$	1	4	9	16	25	36	49	64	$\cdots$
$x^2 \pmod 5) = 0$	1	4	4	1	0	1	4	4	$\cdots$
$x^2 \pmod 7) = 0$	1	4	2	2	4	1	0	1	$\cdots$

where the above pattern is repeated over and over.

(a) From the above table, there are NO squares of the form $5n + 2$.

(b) Again from the above table, x^2 is the form $7n + 2$ whenever x is of the form $7r + 3$ or $7r + 4$, i.e.

$$\begin{aligned} x &= 74+3 && \text{or} \quad 7r+4, \\ 7n+2 = x^2 &= 49r^2+42r+9 && \text{or} \quad 49r^2+56r+16 \\ 7n &= 49r^2+42+7 && \text{or} \quad 49r^2+56r+14 \\ n &= 7r^2+6r+1 && \text{or} \quad 7r^2+8r+2 \end{aligned}$$

where r is an integer.

Q.986 If f is a function, then the notation $f^2(x)$ means $f(f(x))$ and, in general, $f^n(x)$ means $f(\cdots f(x)\cdots)$ where there are n f's. If f is the function

$$f(x) = \frac{x-1}{x+1},$$

find $f^{1000}(3/4)$.

ANS.

$$\begin{aligned} f^2(x) &= \frac{y-1}{y+1} \quad \text{where} \quad y = f(x) = \frac{x-1}{x+1} \\ &= \frac{(x-1)/(x+1)-1}{(x-1)/(x+1)+1} \\ &= \frac{(x-1)-(x+1)}{(x-1)+(x+1)} = -\frac{1}{x} \end{aligned}$$

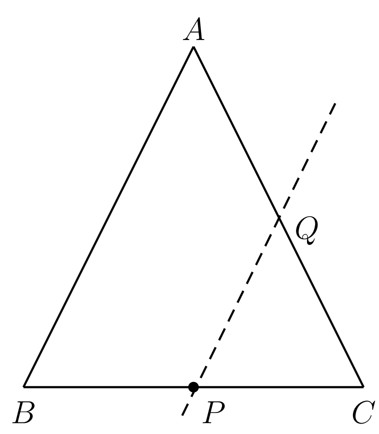
Similarly, if we write g for f^4, then $g(x) = f^4(x) = -\frac{1}{z}$ where $z = f^2(x) = -\frac{1}{x}$

$$= -\frac{1}{-1/x} = x$$

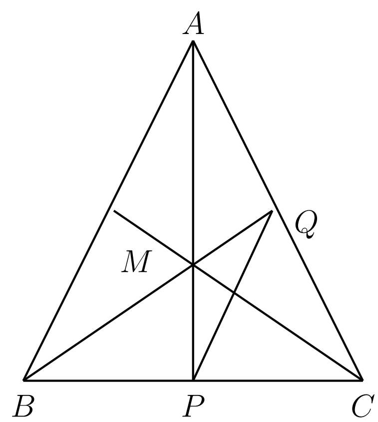
so $f^{1000}(x) = g^{250}(x) = x$.

Q.987 You are in the process of finding the midpoints of the sides of a triangle using a ruler (with no measurements marked on it) and a pair of compasses. However, just as you have constructed one of the midpoints P, you lose your compasses. Fortunately you notice that, if you place your ruler (which has parallel edges) along one of the sides of the triangle, then the point P lies on the opposite edge of the ruler. How would you construct the other two midpoints?

ANS. Suppose that the vertices of the triangle arc A, B, C and that you have constructed the midpoint P of BC :



If the ruler fits exactly between the point P and the side AB, then the side of the ruler through P is parallel to AB, and so passes through the midpoint Q of the other side AC. Since the three medians of a triangle are concurrent, the third midpoint can be found by constructing the intersection M of the lines AP and BQ; then the intersection of CM and AB is the third midpoint:



Q.988 Find all positive integers r, s, t such that r, s, t have no factor in common and

$$\begin{array}{lll} r & \text{divides} & s+t \\ s & \text{divides} & t+r \\ t & \text{divides} & r+s. \end{array}$$

ANS. Write

$$\begin{array}{rcl} s+t &=& rx \\ t+r &=& sy \\ r+s &=& tz \end{array}$$

where x, y, z are three positive integers. We consider two cases:

(a) If none of x, y, z is 1, then $x, y, z \geq 2$ and so $4 \leq 2x$. Thus

$$4r \leq 2rx = 2s + 2t \leq sy + 2t = 3t + r$$

and so $r \leq t$. Similarly $r \leq t \leq s \leq r$ and so $r = s = t$. Since r, s, t have no factor (except 1) in common, $r = s = t = 1$.

(b) If one of x, y, z is 1, then we can suppose that $z = 1$ and so $t = r + s$. Thus

$$\begin{array}{rcl} rx &=& s + t = r + 2s \\ 2s &=& (x-1)r \end{array}$$

similarly, $2r = (y-1)s$ and so

$$\begin{array}{rcl} (x-1)(y-1)rs &=& 4sr \\ (x-1)(y-1) &=& 4. \end{array}$$

If we suppose that $x \geq y$, then

$$x - 1 = y - 1 = 2 \quad \text{or} \quad x - 1 = 4,\ y - 1 = 1$$
$$2s = 2r \quad \text{or} \quad 2s = 4r.$$

Thus $s = r$ or $2r$ and $t = r + s = 2r$ or $3r$. Since r, s, t have no common factors, $r = 1, s = 1, t = 2$ or $r = 1, s = 2, t = 3$.

The only other possibilities are re-arrangements of these values of r, s and t.

Q.989 Suppose a is a real number between 0 and 1. Find all numbers x such that

$$[x] = ax$$

where $[x]$ denotes the greatest integer less than or equal to x (see problem 975).

ANS. Let $x = [x] + y$ where $0 \leq y < 1$. Then the equation can be written

$$\begin{aligned} [x] &= a[x] + ay \\ ay &= (1 - a)[x] \end{aligned}$$

If $a \leq 1/2$, then $1 - a > 1/2 > a$ and so

$$[x] = \frac{a}{1-a} y < y < 1.$$

In this case, the only possibility is $[x] = 0$ and so $a = 0$, when x can be any number between 0 and 1 (except 1).

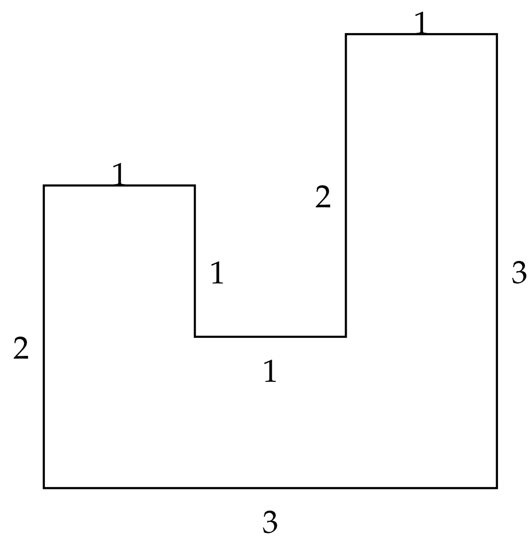
Otherwise, $a > 1 - a$ and

$$[x] = \frac{a}{1-a} y < \frac{a}{1-a}.$$

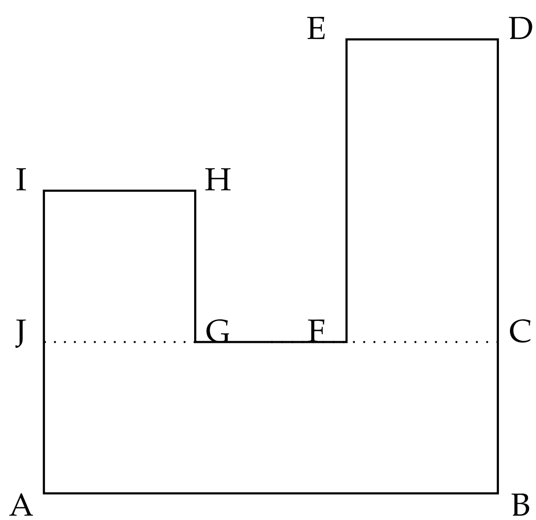
In this case, if r is any non-negative integer less than $a/1 - a$, then there is a solution with $[x] = r$ and so

$$\begin{aligned} y &= \frac{1-a}{a}[x] = \frac{1-a}{a} r \\ x &= [x] + y = r + \frac{1-a}{a} r = \frac{r}{a}. \end{aligned}$$

Q.990 Find the area of the largest 6-sided figure (not necessarily rectangular) which can be drawn inside the 8-sided figure shown.



ANS.



In the figure, all angles are right angles, and

$$IH = GF = ED = 1,\ AI = EF = 2,\ GH = 1.$$

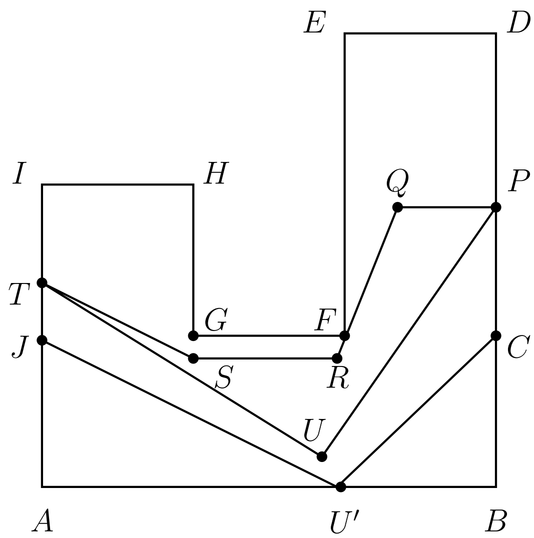
The subregion with vertices $ABDEFJ$ has an area of 5, which is maximum amongst all subregions with six sides. To show this, we consider other subregions with six sides and show that these have area at most 5.

1) If a subregion with six sides has no vertex in the square $GHIJ$, then its area is no greater than that of $ABDEFJA$ i.e., 5.

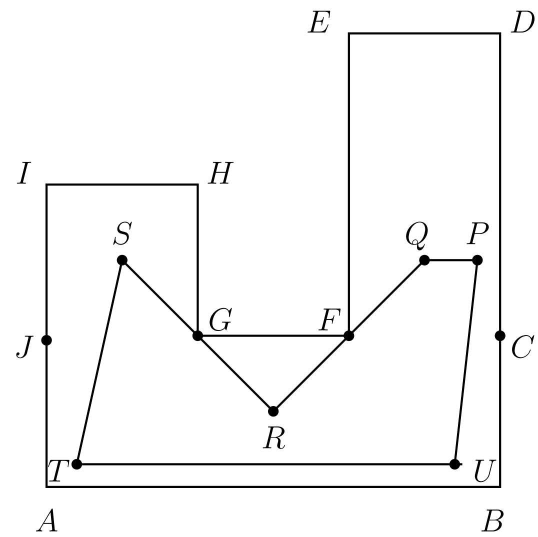
2) If a subregion with six sides has only one vertex in the region $CDEF$, then its total area is at most 5 because the part of the region inside $CDEF$ will be a triangle, of base of most 1 and height at most 2, so of total area at most 1; and the area of the part of the region inside $ABCGHIA$ is at most the area of $ABCGHIA$, viz. 4.

Thus, we need only consider subregions with two (or more) vertices in $CDEF$, and one (or more) vertices in $GHIJ$.

There are several cases to consider:



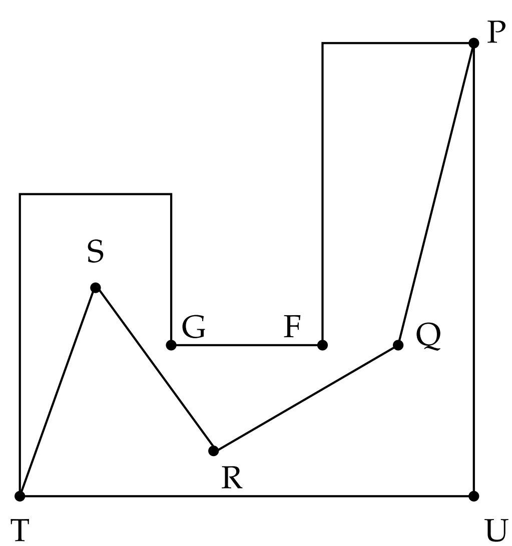
Case I



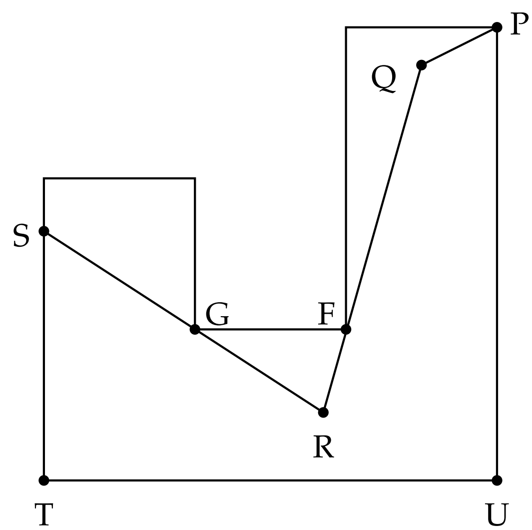
Case II

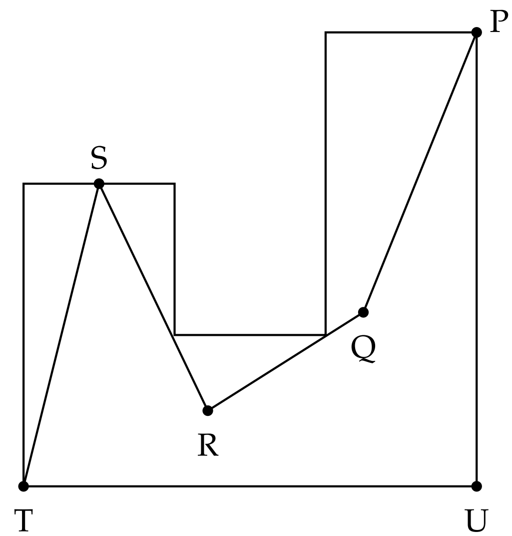
In case I, let U' be the point on AB below U. Then the total area of triangles $U'BC$ and $U'JA$ is $3/2$, and so the maximum area of the figure is at most $6 - 3/2 = 4\frac{1}{2}$. (This case also arises if there are more than three points above JC).

In case II, we obtain a larger area by moving T, U and P to the vertices, thus:



Next, we obtain a larger area by moving S to the edge of the region, and by moving R so that SR and QR pass through G and F.





Continuing in this way, we have to consider just a few basic shapes, all of whom have area at most 5.

Q.991 The letters a, b, c, d, e, f represent the numbers $1, 2, 3, 4, 5, 6$ in some order. It is know that

$$\begin{aligned} a+b &< c+d \\ \text{and} \qquad c+e &< a < f. \end{aligned}$$

Find the values of a, b, c, d, e, f (in order).

ANS. First note that $c+e+b < a+b < c+d$ and so

$$b+e < d.$$

Also, since the smallest sum of any two of the numbers b, c, e is 3,

$$\begin{aligned} 3 &\leq c+e < a < f \leq 6 \\ \text{and} \qquad 3 &\leq b+e < d. \end{aligned}$$

so $a = 4$ or 5 and $d = 4, 5$ or 6.

If $a = 5$, then $f = 6$ and so $d = 4$, and $b, c, e \in \{1, 2, 3\}$ with

$$\begin{aligned} 5+b &= a+b < c+d = c+4 \\ c+e &< a = 5 \\ b+e &< d = 4. \end{aligned}$$

It is not hard to show that no arrangement of the numbers satisfies all these inequalities.

So $a = 4, \{d, f\} = \{5, 6\}$ and $\{c, e\} = \{1, 2\}$. This means that $b = 3$, and so

$$7 = a + b < c + d \leq 2 + 6 = 8.$$

The only possibility is $c + d = 8$ and so $c = 2,\ d = 6,\ e = 1$ and $f = 5$.

Q.992 In a given (co-ed) school, each boy has gone out with at least one girl (but not every girl) and each girl has gone out with at least one boy (but not every boy). Show that there are two boys B, and B_2 and two girls G_1 and G_2 such that each of B_1 and B_2 has gone out with exactly one of G_1 and G_2, and each of G_1 and G_2 has gone out with exactly on of B_1 and B_2.

ANS. Choose a boy B_1 who has gone out with as many girls as possible. Since B_1 has not gone out with every girl, there is a girl G_2 who has not gone out with B_1. Since every girl has gone out with at least one boy, there is a boy B_2 who has gone out with G_2.

The number of girls with whom B_2 has gone out is not greater than the number of girls with whom B_1 has gone out, and in addition B_2 has gone out with G_2, though B_1 has not. Thus, amongst all the girls with whom B_1 has gone out, there must be at least one, G_1 say, with whom B_2 has not gone out, so each of B_1 and B_2 has gone out with exactly one of G_1 and G_2, and each of G_1 and G_2 has gone out with exactly one of B_1 and B_2.

THE FIVE PIECE CHESS ENDGAME

Peter Donovan[1], with acknowledgements to www.chess-space.com/Endings/[2]

Remarkably it is now possible to obtain complete analyses of chess positions with up to five pieces. This has revealed that the 50 move rule [the player to move may claim a draw if none of the previous 50 moves by either player have been captures or pawn moves] is not based on any scientific rationale. Indeed A. Troitsky showed early this century that the endgame King + Knight + Knight versus King + Pawn sometimes needs an extension of the 50 move rule. If the 50 move rule is replaced by a 200 move rule some positions that were previously drawn become wins for the superior side. A good reference is John Nunn's 'Secrets of Pawnless Endgames'.

1. Bg3 Nc4	2. Kd1 Ne3+	3. Ke2 Nf5	4. Bf2 Ne7	5. Kf3 Nc6	6. Kg3 Nd4
7. Kg4 Nb3*	8. Kh3 Nc1*	9. Bb6* Nd3	10. Nc2 Nf2+	11. Kg3 Ne4+	12. Kf4 Nd2
13. Ne3 Kh2	14. Kg4 Nb3	15. Ba7 Nc1	16. Bd4 Nd3	17. Kh4 Nf2	18. Bb6* Ne4
19. Ba5 Kg1	20. Be1 Kh1*	21. Kg4* Kg1	22. Kf4 Nc5	23. Kf3 Kh2*	24. Ng4+ Kh3
25. Nf2+ Kh2	26. Ba5 Ne6	27. Bb6 Nf8	28. Bc7+ Kg1	29. Ng4 Ne6	30. Bb6+ Kf1
31. Ne3+ Kg1	32. Kg3 Nf8	33. Bc5* Ne6	34. Ba7 Nd8	35. Bb6 Nc6	36. Kf3 Kh2
37. Ng4+ Kh1	38. Bc5* Nd4+	39. Kg3 Nb3	40. Be3 Na5	41. Kf2 Nc4	42. Bh6 Nd2*
43. Nf6 Nb3	44. Ne4 Kh2	45. Ng5 Nc5	46. Bf8 Nd3+	47. Ke3 Nb2	48. Bg7 Nc4+
49. Kf2 Kh1*	50. Bd4 Nd6	51. Kf3 Nf5	52. Ba7* Ne7	53. Bc5 Nd5	54. Ne4* Kh2
55. Nf2 Nc3	56. Bd6+ Kg1	57. Ng4* Nb5	58. Bc5+ Kh1	59. Nf2+ Kg1*	60. Nd3+* Kh1
61. Nf4 Nc3	62. Bb6* Kh2	63. Bc7* Kh1*	64. Be5 Nb5	65. Ke4 Na7	66. Bc7* Nb5*
67. Bb8 Kg1	68. Kd3* Kf2	69. Nd5 Kf3	70. Nb4 Na3	71. Nc6 Nb5	72. Na5 Kf2*
73. Kc4 Na3+	74. Kb3 Nb1	75. Nc4 Kf3	76. Ba7* Ke4*	77. Kc2 Kd5	78. Ne3+ Kc6*
79. Kxb1 Kb5*	80. Bd4* Ka4*	81. Kc2 Ka5*	82. Kd3* Ka6*	83. Kc4* Kb7	84. Kb5* Kc7
85. Ba1* Kb7*	86. Nd5 Ka7*	87. Be5* Ka8*	88. Kc6 Ka7	89. Nb6 Ka6	90. Bb8 Ka5
91. Nd5 Ka4	92. Be5* Ka5*	93. Bd4 Ka6	94. Nb4+* Ka5	95. Kc5 Ka4	96. Kc4 Ka5
97. Ba7* Ka4	98. Bb6 Ka3	99. Nd3 Ka4	100. Nb2+* Ka3	101. Kc3 Ka2	102. Bc5* Ka1*
103. Kb3* Kb1	104. Be3* Ka1	105. Nc4 Kb1	106. Na3+ Ka1	107. Bd4×	

[1]Peter Donovan is a Pure Mathematician at the University of N.S.W.

[2]Editorial note, February 2014: this is a dead link

Computer analysis has shown that many endgames with White having King + Bishop + Knight versus King + Knight are dvwins for the White side. In fact 32% of these positions are wins for White. All of those with White to move that can be won at all can be won in 107 moves. There are a few where this maximum is needed, such as the following:

White is to move in the position: wKc1 wBh2 wNe1 bKh1 bNb2.

Best play for both sides is set out in the above table. An asterisk indicates the existence of alternative moves which are of equal merit.

Answer to Safety First

$$\begin{array}{r} 8\,9\,5\,7 \\ 4\,9\,6\,7 \\ \hline 1\,3\,9\,2\,4 \\ \hline \end{array} \qquad \text{or} \qquad \begin{array}{r} 8\,9\,6\,7 \\ 4\,9\,5\,7 \\ \hline 1\,3\,9\,2\,4 \\ \hline \end{array}$$