Parabola Volume 33, Issue 2 (1997)

The Power and the Polynomial.

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Some years ago I saw the following problem in a mathematics competition.

Solve the simultaneous equations

$$\begin{cases} x+y+z=1 \\ x^2+y^2+z^2=29 \\ x^3+y^3+z^3=-29 \end{cases}$$

If you're game you can try to solve this problem and you might even guess a solution, although proving it is the only solution is a little harder. Certainly attempting to eliminate two of the unknowns would be a foolhardy thing to do. A seemingly unrelated problem, but one which we shall see involves similar ideas is:
Let a, b, c be real numbers such that $a+b+c=0$. Prove that

$$\frac{(a^5+b^5+c^5)}{5} = \frac{(a^3+b^3+c^3)}{3}.\frac{(a^2+b^2+c^2)}{2}.$$

Both of these problems involve expressions which are **symmetrical**. That is, if we interchange any of the symbols, the expressions remain the same. The key to both these problems is to look at a cubic equation which has roots x, y, z in the first case, and roots a, b, c in the second.

If we construct monic polynomials of degree 2,3,4,.. with given roots $\alpha, \beta, \gamma, ...$ we can begin to see some useful patterns.

For quadratics,

$$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$$

and so if α, β are the roots of the quadratic $ax^2+bx+c=0$ then

$$\alpha+\beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

For cubics,

$$(x-\alpha)(x-\beta)(x-\gamma) = x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)x - \alpha\beta\gamma$$

and so if α, β, γ are the roots of the cubic $ax^3+bx^2+cx+d=0$ then

$$\alpha+\beta+\gamma = -\frac{b}{a}, \quad \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}, \quad \alpha\beta\gamma = -\frac{d}{a}.$$

For quartics

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) =$$

$$x^4-(\alpha+\beta+\gamma+\delta)x^3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x+\alpha\beta\gamma\delta,$$

and so if $\alpha, \beta, \gamma, \delta$ are the roots of the cubic $ax^4 + bx^3 + cx^2 + dx + e = 0$ then

$$\alpha + \beta + \gamma\delta = -\frac{b}{a}, \qquad \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{c}{a}$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}, \qquad \alpha\beta\gamma\delta = \frac{e}{a}.$$

You can see how the pattern continues.

Examples:

1. If $p(x) = 3x^3 - 2x^2 + x - 1$ has roots α, β, γ, then
$\alpha + \beta + \gamma = \frac{2}{3}, \quad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{1}{3}, \quad \alpha\beta\gamma = \frac{1}{3}$

2. If $q(x) = 2x^4 - 3x^3 + x^2 + 5x - 7$ has roots $\alpha, \beta, \gamma, \delta$, then

$$\alpha + \beta + \gamma + \delta = \frac{3}{2}, \quad \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{1}{2}$$

$$\alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-5}{2}, \quad \alpha\beta\gamma\delta = \frac{-7}{2}.$$

These formulae can be used to solve the first of the posed problems. But before we do that, let's have a look a similar but slightly easier one.

3. Solve the simultaneous equations

$$\begin{cases} a+b+c=2 \\ bc+ac+ab=-5 \\ abc=-6 \end{cases}$$

Suppose a, b, c are roots of a cubic equation, then we are given here the co-efficients of that cubic. In fact a, b, c satisfy, $x^3 - 2x^2 - 5x + 6 = 0$ which has an obvious root of $x = 1$, and so the cubic factors as $(x-1)(x-3)(x+2) = 0$. Hence the solution is $a, b, c = 1, 3, -2$ in any order.

Observe that the formulae above are all **symmetric**, i.e. they remain the same when the variables are re-arranged. It turns out that **any** symmetric expression involving the roots of a polynomial can be evaluated in terms of the co-efficients of the polynomial. I will not prove this here, but some examples will show the result to be at least plausible. Newton gave the first proof of this.

Example:

4. If α, β are the roots of $2x^2 - 5x + 4 = 0$, find $\alpha^2 + \beta^2$ and $\frac{1}{\alpha} + \frac{1}{\beta}$.

Firstly, $\alpha+\beta=\frac{5}{2}$ and $\alpha\beta=2$

Now $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{9}{4}$ and $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{5}{4}$.

Observe that in both cases we could re-arrange the symmetric expression and write it in terms of the sum and product of the roots.

Example:
5. If α, β, γ are the roots of $x^3-7x^2+2x+4=0$ find $\alpha^2+\beta^2+\gamma^2$ and $\alpha^2\beta+\beta^2\alpha+\alpha^2\gamma+\beta^2\gamma+\gamma^2\beta+\gamma^2\alpha$.

Firstly $\alpha+\beta+\gamma=7, \quad \alpha\beta+\alpha\gamma+\beta\gamma=2, \quad$ and $\alpha\beta\gamma=-4$,
and $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)=49-2\times 2=45$

The second expression to be found is $(\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha+\beta+\gamma)-3\alpha\beta\gamma=2\times 7-3\times(-4)=26$.

You should now try to solve the simultaneous equations

$$\begin{cases} a+b+c=-1 \\ a^2+b^2+c^2=9 \\ abc=4. \end{cases}$$

By the way, up until this point, the ideas I have talked about are all from 3-Unit and 4-Unit High School mathematics.
To properly solve the initial problems I posed we need to go a little beyond the High School syllabus and find some nice formulae for the **sums of powers of the roots**.

Suppose α, β, γ are the roots of $x^3+a_1x^2+a_2x+a_3=0$. We will let s_n be the sum of the nth powers of the roots. We can thus write down the following results:

$$s_1=\alpha+\beta+\gamma=-a_1 \text{ and so } s_1+a_1=0$$

$$s_2=\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)=a_1^2-2a_2$$

I will write this as

$$s_2+s_1a_1+2a_2=0.$$

After some hard work, you can show that the next line in the pattern is

$$s_3+s_2a_1+s_1a_2+3a_3=0.$$

After this, the pattern changes a little and becomes,

$$s_4+s_3a_1+s_2a_2+s_1a_3=0$$

and

$$s_5 + s_4a_1 + s_3a_2 + s_2a_3 = 0$$

and so on.

Starting with s_1, we can build up to any power we like.

These formulae are all we need to solve our problems. but the full story is:

Suppose $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of $x^n + a_1x^{n-1} + .. + a_n = 0$.

$$s_1 + a_1 = 0$$

$$s_2 + a_1s_1 + 2a_2 = 0$$

$$s_3 + a_1s_2 + a_2s_1 + 3a_3 = 0$$

$$s_4 + a_1s_3 + a_2s_2 + a_3s_1 + 4a_4 = 0$$

....

$$s_{n-1} + a_1s_{n-2} + ... + (n-1)a_{n-1} = 0$$

$$s_n + a_1s_{n-1} + ... + na_n = 0$$

The pattern in fact continues beyond here,

$$s_{n+1} + a_1s_n + ... + s_1a_n = 0$$

$$s_{n+2} + a_1s_{n+1} + ... + s_2a_n = 0$$

....

These were also proved by Newton.

Some examples are in order.

Example:
6. Find the sum of the cubes and the sum of the fourth powers of the roots of the equation $x^3 - 2x^2 + 3x + 1 = 0$.

$s_1 = 2, \quad s_2 - 2s_1 + 2 \times 3 = 0$ and so $s_2 = -2$
$s_3 - 2s_2 + 3s_1 + 3 \times 1 = 0$ and so $s_3 = -13$.
$s_4 - 2s_3 + 3s_2 + s_1 = 0$ and so $s_4 = -22$.

We can now (finally) solve the first problem posed. Solve the simultaneous equations

$$\begin{cases} x + y + z = 1 \\ x^2 + y^2 + z^2 = 29 \\ x^3 + y^3 + z^3 = -29 \end{cases}$$

Let x, y, z be the roots of the cubic $w^3 + a_1 w^2 + a_2 w + a_3$. The first equation gives $a_1 = -1$, and since $s_2 + s_1 a_1 + 2a_2 = 0$ we have $29 - 1 + 2a_2 = 0$ and so $a_2 = -14$. Also $s_3 + s_2 a_1 + s_1 a_2 + 3a_3 = 0$ and so $a_3 = 24$. Thus, the numbers we seek are the roots of the cubic equation $x^3 - x^2 - 14x + 24 = 0$. This we can solve by trial and error and obtain the solutions $2, 3 - 4$ and so $x, y, z = 2, 3, -4$ in any order.

For our second problem:

Let a, b, c be real numbers such that $a + b + c = 0$. Prove that

$$\frac{(a^5 + b^5 + c^5)}{5} = \frac{(a^3 + b^3 + c^3)}{3} . \frac{(a^2 + b^2 + c^2)}{2} .$$

Let a, b, c be the roots of $x^3 + a_1 x^2 + a_2 x + \gamma = 0$ and note that $a_1 = a + b + c = 0$. Thus, $s_1 = 0$, and $s_2 = -2a_2$, $s_3 + a_2 s_1 + 3a_3 = 0$ so $s_3 = -3a_3$. Also $s_4 + a_2 s_2 + a_3 s_1 = 0$ and so $s_4 = 2a_2^2$ and finally, $s_5 + a_2 s_3 + a_3 s_2 = 0$ giving $s_5 = 5a_2 a_3$. Thus $\frac{s_3}{3}\frac{s_2}{2} = a_2 a_3 = \frac{s_5}{5}$ and the result is proven.

Here is a challenge for you to try. The problem was first posed last century.

If $x + y + z = 0$ show that

$$\frac{(x^{11} + y^{11} + z^{11})}{11} = \frac{(x^3 + y^3 + z^3)}{3} . \frac{(x^8 + y^8 + z^8)}{2} - \frac{(x^3 + y^3 + z^3)^3}{9} . \frac{(x^2 + y^2 + z^2)}{2} .$$