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THE PROBLEM THAT (MAYBE?) FOOLED JOHN VON NEUMANN

S M Stewart[1]

In an interesting problem, which as my title suggests also has interesting historical roots, physical insight can often simplify an otherwise complicated mathematical problem. It involves a problem proposed to John von Neumann (Hungarian born American mathematician 1903–1957) by his friend and colleague Stanislaw Ulam (Polish born American mathematician 1909–1984) while the two were travelling on a train together. It went something like the following:

> Two objects A and B are separated by a distance L. At the same instant, both objects begin to move towards each other at a common constant speed of v. Also at the same instant, a third object C leaves A and heads towards B at a constant speed of u such that $u > v$. Upon object C meeting B it turns around (instantaneously) and heads back towards A. Such a process is repeated until objects A and B collide. The question is, what will be the total distance covered by object C?

Physical approach

The problem can be solved most readily, with a little physical insight, by employing the simple formula for constant speed we are all familiar with

$$\text{speed} = \frac{\text{distance}}{\text{time}} \quad .$$

By recognising that objects A and B collide after having travelled a distance of $L/2$, the time, t, at which the collision occurs at will be given by

$$t = \frac{L}{2v} \quad . \quad (\star)$$

Thus the distance, d, covered by object C in this time, as given by $(\star)$, will just be

$$d = ut = \frac{Lu}{2v} \quad .$$

Of course, when such a problem was proposed to von Neumann he solved it immediately leading Ulam to reply 'So you saw the trick' (namely that of the above approach). Von Neumann replied 'Trick? What trick? I just summed the series.' [1]. So what series did von Neumann have in mind, and where does it come from?

[1] PhD student, Department of Physics, University of Wollongong

Mathematical approach

If the preceding physical trick is not seen, as it was supposed with von Neumann, the power and versatility of mathematics can always be relied upon. Consider the following figure which plots distance versus time for each of the three objects.

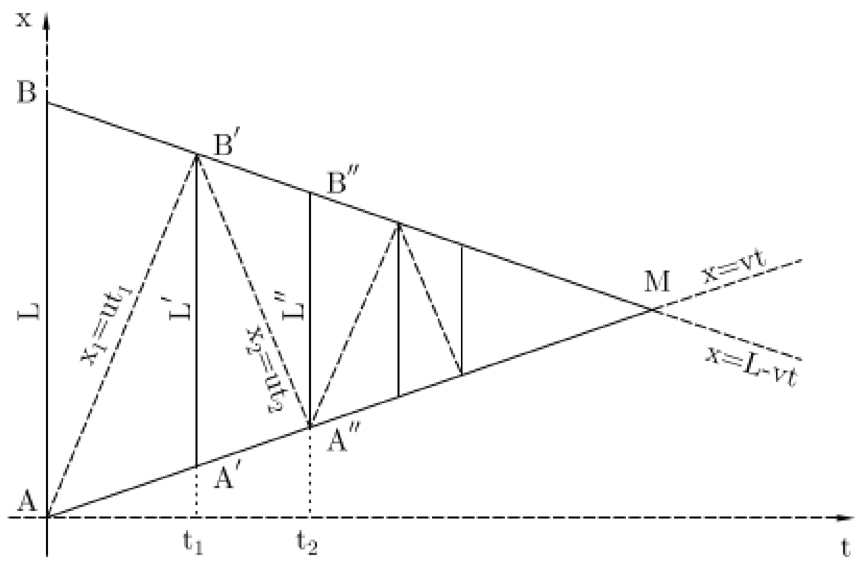

Figure 1: Illustrating how.

AB, $A'B'$, $A''B''$, etc. represent the separation distances between objects A and B at the instant when object C coincides with either one of these objects. From figure 1 $AB \| A'B' \| A''B'' \| ...$ Hence $\angle ABM = \angle A'B'M = \angle A''B''M = ...$ and $\angle BAM = \angle B'A'M = \angle B''A''M = ...$ (alternate angles between parallel lines are equal). Since $\angle M$ is a common angle, the triangles ABM, $A'B'M$, $A''B''M$, etc. are all similar triangles (equiangular). Since corresponding sides in similar triangles are of the same ratio, let us define the scaling factor, q, as

$$q = \frac{A'B'}{AB}\left(= \frac{L'}{L}\right) = \frac{A''B''}{A'B'}\left(= \frac{L''}{L'}\right) = ...$$

The total distance travelled by object C will therefore be given by the following sum

$$d = x_1 + x_2 + x_3 + ... \tag{1}$$

where the x_i's are the distances travelled by object C between each successive turn. But since object C moves at constant speed u, $x_i = ut_i$. Thus (1) becomes

$$d = u[t_1 + t_2 + t_3 + ...] \tag{2}$$

where the t_i's now correspond to the time intervals between each successive turn. Since object C makes infinitely many turns, the solution to the problem thus depends on an infinite series being summed, which it is assumed may be very complicated. I will now show that this is in fact not the case, as von Neumann recognised.

Consider the first (i.e. $i = 1$) turn when objects A and B are a distance of L apart. In the time, t_1, it takes object C to reach object B from object A, object C will have travelled a distance of $x_1 = ut_1$ while object B will have travelled a distance of $x_B = vt_1$. Since

$$L = x_1 + x_B = (u + v)t_1$$

then

$$t_1 = \frac{L}{u+v} \quad . \tag{3}$$

Also in this time, by symmetry, objects A and B have come closer together by a distance of

$$X' = 2x_B = 2vt_1 = \frac{2vL}{u+v} \quad .$$

The new separation distance between objects A and B is now

$$L' = L - X' = \frac{u-v}{u+v}L \quad .$$

We can now find the scaling factor as

$$q = \frac{u-v}{u+v} \quad . \quad (\star\star)$$

Now consider the second (i.e. $i = 2$) turn when objects A and B are a distance of L' apart. In the time, t_2, it now takes object C to reach object A again, object C will have travelled a distance of $x_2 = ut_2$ while object A will have travelled a distance of $x'_A = vt_2$. Since

$$L' = x_2 + x'_A = (u + v)t_2$$

from which

$$t_2 = \frac{L'}{u+v} = \frac{qL}{u+v} \quad . \tag{4}$$

The whole situation is now exactly as it was at the beginning except it has been rescaled by the scaling factor. This process is then to be repeated over and over again ad infinitum. It is clear that an infinite series for the total distance travelled by object C develops where the time taken to traverse the i^{th} turn is just $t_i = q^i L/(u + v)$. From (2), (3), (4), etc. one finds

$$d = \frac{Lu}{u+v}[1 + q + q^2 + ...] \quad . \tag{5}$$

The sum appearing in (5) is just an infinite geometric series which can be summed to give

$$1 + q + q^2 + ... = \frac{1}{1-q} = \frac{u+v}{2v}$$

where use of ($\star\star$) has been made. Combining this with (5) one obtains

$$d = \frac{Lu}{2v}$$

as before! Thus it appears von Neumann was not so easily fooled.

Reference

[1] *Mathematics Teacher*, Vol. **84**, No. 8, November 1991, p. 636.