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HOW TO CONSTRUCT REGULAR 7-SIDED POLYGONS — AND MUCH ELSE BESIDES

Part 2 — Some New Mathematics

by

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Introduction

In Part 1 (Parabola Vol. 34, No 1) we introduced you to a basic construction whereby we folded down m times at the top of a tape and folded up n times at the bottom of the tape (see Figure 1). Such a procedure is called a **period-2 folding procedure**, more specifically, the **(m, n)-folding procedure**. In fact, we only discussed the special cases

$$(m, n) = (1, 1),\ (2, 2),\ (3, 3),\ (2, 1)$$

but it surely must have been clear that we could have carried out the basic construction for any positive integers m, n. We discuss here what we would have got, in general.

Suppose the angle appearing at the top of the tape at the k^{th} stage is u_k; and the angle appearing at the bottom of the tape at the k^{th} stage is v_k (see Figure 1). Then, summing the angles, we get at the bottom of the tape at the k^{th} stage (remember we are using radian measure),

$$u_k + 2^n v_k = \pi$$

and at the top of the tape at the $(k+1)^{\text{st}}$ stage

$$v_k + 2^m u_{k+1} = \pi$$

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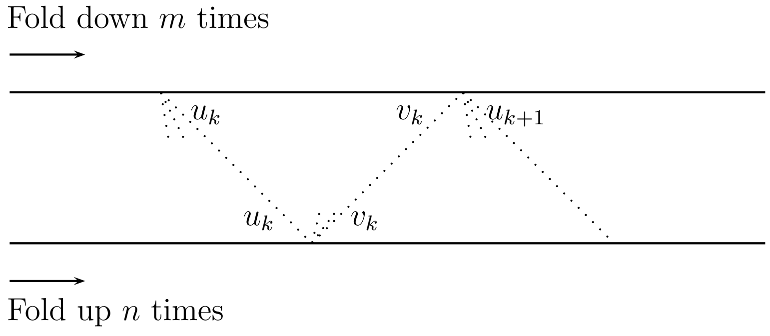

Figure 1: kth stage (period-2 folding)

so that

$$u_k + 2^n(\pi - 2^m u_{k+1}) = \pi$$

or

$$2^{m+n} u_{k+1} - u_k = (2^n - 1)\pi \tag{1}$$

Thus the successive angles at the top of the tape, namely,

$$u_0,\ u_1,\ u_2, \cdots, u_k,\ u_{k+1},\ \cdots$$

are related by equation 1 and we ask the question — how do these angles behave as k gets large? (We saw in Part 1 what happened in the special case $m = 2$, $n = 1$; but this case wasn't really very special!)

In fact, to answer our question, we will make the problem even more general! Suppose successive numbers

$$u_0,\ u_1,\ u_2, \cdots, u_k,\ u_{k+1},\ \cdots$$

are related by

$$a u_{k+1} = u_k + b, \text{ with } a > 1. \tag{2}$$

We will show that then u_k gets closer and closer to $\overline{u}$, which is the solution to the associated equation

$$au = u + b, \tag{3}$$

that is

$$\overline{u} = \frac{b}{a-1}. \tag{4}$$

For let us set $e_k = u_k - \overline{u}$. Then $u_k = e_k + \overline{u}$, so equation 2 becomes

$$a(e_{k+1} + \overline{u}) = (e_k + \overline{u}) + b$$

or

$$ae_{k+1} = e_k, \quad \text{since} \quad a\overline{u} = \overline{u} + b. \tag{5}$$

How, then, does the sequence $e_0,\ e_1,\ \cdots,\ e_k,\ e_{k+1},\ \cdots$ behave? We have

$$e_1 = \frac{1}{a}e_0$$
$$e_2 = \frac{1}{a}e_1 = \frac{1}{a^2}e_0$$
$$e_3 = \frac{1}{a}e_2 = \frac{1}{a^3}e_0$$

and, in general,

$$e_k = \frac{1}{a^k}e_0$$

But, since $a > 1$, it follows that $\frac{1}{a^k}$ gets smaller and smaller as k increases so that, as we say, e_k (the error at the k^{th} stage) tends to 0 as k tends to infinity. Thus u_k tends to $\overline{u} = \frac{b}{a-1}$. It is important for our paper-folding procedure to note that the value of $\overline{u}$ is quite independent of the initial error e_0. It is determined by the attractive device of 'ignoring the suffixes on the u's'! Thus, reverting to (1.1), we find that the angle at the top of the tape, u_k, tends to

$$\overline{u} = \frac{2^n - 1}{2^{m+n} - 1}\pi.$$

Notice that all this agrees with what we found in Part 1 in the case $m = 2,\ n = 1$: for then $\overline{u} = \frac{\pi}{7}$.

Now suppose that (as in the case $m = 2,\ n = 1$) $\frac{2^{m+n}-1}{2^n-1}$ happens to be an integer, say, $s = \frac{2^{m+n}-1}{2^n-1}$. Then by using the (m, n)-folding procedure, followed by the FAT algorithm, we can fold a regular s- gon. Thus we have the following fundamental questions:

Question 1: When is $\dfrac{2^{m+n}-1}{2^n-1}$ an integer?

Question 2: How do we recognize that an integer s is of the form $\dfrac{2^{m+n}-1}{2^n-1}$?

Question 3: If the integer s is of this form, how do we determine m and n as functions of s?

We answer these questions in Section 2; and we make some further developments of these mathematical ideas in Section 3. However, it is plain to see, from elementary algebra, that if $n = m$ then $s = \frac{2^{2n}-1}{2^n-1} = 2^n + 1$. Thus (as you may have guessed from doing the three experiments at the end of Part 1) the D^nU^n procedure (see Figure 2) produces tape on which the smallest angle u_k approaches $\dfrac{\pi}{2^n+1}$. It is a special feature of this tape that you can use it to produce regular (2^n+1)-gons not only by the FAT algorithm, but also by folding on consecutive crease lines of the same length (and there will be n different lengths from which to choose).

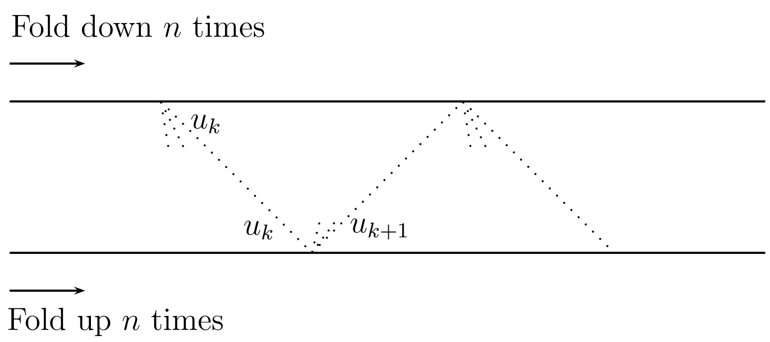

Figure 2: kth stage (period-1 folding)

Folding Numbers

We call an integer of the form $\dfrac{2^{m+n}-1}{2^n-1}$, where m, n are positive integers, a folding number. We know from the argument in the Introduction that we can fold paper to produce a regular s-gon, using the (m,n) folding procedure and the FAT algorithm, if s has the given form.

However, from the number-theoretical point of view, there is one feature of our set of three questions at the end of the Introduction which introduces an irrelevant restriction into our investigation. This is the special role assigned to the number 2: this role, as is easily seen, arises because we — very naturally and properly! — confine ourselves to bisecting angles by paper-folding. However, the arithmetical questions we have raised make perfect sense — and should not be harder to answer — if we replace 2 by an arbitrary positive integer $t \geq 2$. Thus we look at integers s of the form

$$s = \frac{t^{m+n} - 1}{t^n - 1}, \quad \text{for a fixed integer} \quad t \geq 2. \tag{6}$$

We call such integers $\boldsymbol{t}$-folding numbers; we usually suppress the t if $t = 2$.

Next, it is not convenient to have the expression $(m + n)$ appearing in (2.1); obviously, we are concerned with integers s of the form

$$\frac{t^a - 1}{t^b - 1}, \quad \text{where } a, b \text{ are positive integers with } a > b, \tag{7}$$

Thus, to answer Question 1, we must find out when $t^b - 1$ is a factor of t^a-1. From our knowledge of elementary algebra we should be able to say immediately that $t^b - 1$ is a factor of $t^a - 1$ if b is a factor of a. However, it actually turns out that this sufficient condition is also necessary, that is, we have the following theorem.

Theorem 0.1. $t^b - 1$ *is a factor of* $t^a - 1$ *if and only if* b *is a factor of* a.

In fact, a fairly easy argument shows that the gcd (greatest common divisor) of $t^b - 1$ and $t^a - 1$ is $t^d - 1$ where $d = \gcd(a, b)$; and Theorem 2.1 follows quickly from this. Notice that the answer to the question of whether $t^b - 1$ is a factor of $t^a - 1$ actually does not depend on the value of t — an immediate justification for broadening the scope of our arithmetical investigation.

We turn now to Question 2. We will suppose then that we are dealing with an integer s, where s has the form (2.1). Then we know by Theorem 2.1 that n is a factor of m. Moreover, the quotient $\frac{m+n}{n} \geq 2$. Since, in the form (2.2), b is a factor of a, we prefer to write

$$b = x, \;\; a = xy, \;\; x, y \text{ positive integers with } y \geq 2. \tag{8}$$

y	$2'' - 1$													
26	67708863													
25	33554431													
24	16777215													
23	8389607													
22	4194303													
21	2097151													
20	1048575													
19	524287													
18	262143													
17	131071													
16	65535													
15	32767													
14	16383													
13	8191	22369621												
12	4095	5592405												
11	2047	1398101												
10	1023	349525												
9	511	87381	19173961											
8	255	21845	2396745											
7	127	5461	299593	17895697										
6	63	1365	37449	1118481	34636833									
5	31	341	4681	69905	1082401	17043521								
4	15	85	585	4369	33825	266305	2113665	16843009						
3	7	21	73	273	1057	4161	16513	65793	26257	1049601	4196353	16781313	67117057	
2	3	5	9	17	33	65	129	257	513	1025	2049	4097	8193	$2^r + 1$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
y / x	1	1	3	4	5	6	7	8	9	10	11	12	13	$14 \leq x \leq 26$

Table 1: 2-folding numbers – the number in position (x, y) is $\frac{2^{xy} - 1}{2^x - 1}$.

Thus s has the form (2.2), with a, b given by (2.3). We propose to write s in base t. We know, of course, that the numerical form for s is then unique. Now it is easy to see that

$$\frac{t^{xy}-1}{t^x-1} \overset{(t)}{=} \underbrace{10\cdots0}\ \underbrace{10\cdots0}\ \cdots\ \underbrace{10\cdots0}\ 1 \tag{9}$$

where $\overset{(t)}{=}$ means that the expression to the right is written in base t; where the 'repeating part' $\underbrace{10\cdots0}$ consists of 1 followed by $(x-1)$ zeros; and where there are y 1's. Table 1 shows some of the values of 2-folding numbers. It may be of interest to you to write some of these numbers in base 2 just to see how they fit the form of (2.4). We have then, in fact, answered Questions 2 and 3. For a positive integer s is a $\boldsymbol{t}$-folding number if and only if it assumes the form on the right of (2.4) when written in base t and then x, y are determined by s. But then m and n are determined from x and y by the rule

$$n = x, \qquad m + n = xy. \tag{10}$$

Example 2.1 Is 85 a folding number? If so, how do we fold a regular 85-gon? If we didn't have Table 1 available we could proceed as follows. First write 85 in base 2 obtaining $85 \overset{(2)}{=} 1010101$. Thus $x = 2$, $y = 4$, so $n = 2$, $m + n = 8$, $m = 6$. Hence 85 is a folding number and we may fold a regular 85-gon by the $(6, 2)$ folding procedure, followed by the FAT algorithm.

Example 2.2 Is 757 a 3-folding number?
Now $757 \overset{(3)}{=} 1001001$. Thus $x = 3$, $y = 3$, so $757 = \frac{3^9-1}{3^3-1}$.

Example 2.3 Is 13 a folding number?
Now $13 \overset{(2)}{=} 1101$ and so is not a folding number (there is no 'repeating part'). It is easy to see, however, that 13 is a 3-folding number.

Folding other Polygons

We close this article by discussing one way to fold regular a-gons where the odd number a is not a folding number; for example (see above, Example 2.3), how might we fold a 13-gon? The solution we give here is not as pretty as that which will form a principal part of our next article, but, on the other hand, it is much easier to understand than the method to be described in the next article.

Once again, however, we work with base t rather than confining ourselves to the case $t = 2$ thrown into prominence by our paper- folding procedures. Thus the condition that a be odd is replaced by the condition that a be prime to t, that is, that $\gcd(a, t) = 1$. We then claim that, given any x, there exists y such that

$$a \text{ divides } \frac{t^{xy} - 1}{t^x - 1}, \tag{11}$$

indeed, the set of such y is the set of multiples of a basic y_0, which we call the $\boldsymbol{x}$-height of a. Look at Table 1 (of 2-folding numbers) to understand this terminology.

Our claim is based on the fact that t is prime to $a(t^x - 1)$, from which it follows that there exists a positive integer z_0 such that $t^z - 1$ is divisible by $a(t^x - 1)$ if and only if z is a multiple of z_0. The argument is then completed by invoking Theorem 2.1 which tells us that, for $\frac{t^{z_0}-1}{t^x-1}$ to be an integer, we must have $z_0 = xy_0$ for some y_0.

Let $h = h(a)$ be the **1**-height (or, more simply, the height) of a. This means that h is the smallest positive integer such that[3] $a \Big| \frac{t^h-1}{t-1}$. It is then not difficult to see that if $a \Big| \frac{t^{xy}-1}{t^x-1}$, then $h|xy$. This result leads to the following rather surprising conclusion.

We would have two obvious criteria for choosing the most convenient t-folding number $s = \frac{t^{xy}-1}{t^x-1}$, such that $a|s$. The paper folder would like to minimize xy, which we may call the total number of folds — remember that $n = x$, $m+n = xy$. The mathematician would like to minimize s, so that the process of passing from a regular s-gon to a regular a-gon is as simple as possible. Now it is by no means obvious that those two criteria lead to the same choice of s; after all, we can have two folding numbers s_1, s_2 with $s_1 > s_2$ but s_1 requires fewer folds than s_2. Thus

$$s_1 = \frac{2^5 - 1}{2^1 - 1} = 31, \quad s_2 = \frac{2^6 - 1}{2^3 - 1} = 9,$$

[3] Here we are using the notation $A\Big|B$ to indicate that the integer A divides the integer B.

so $s_1 > s_2$, but an s_1-gon requires 5 folds where an s_2-gon requires 6 folds.

Remarkably, however, if we confine attention to those t-folding numbers which have a as a factor, we find the following

Theorem 0.2. *If $s = \frac{t^{xy}-1}{t^x-1}$ is the smallest t-folding number having a as a factor then $h(a) = xy$.*

Corollary 0.3. *The smallest t-folding number $s = \frac{t^{xy}-1}{t^x-1}$ having a as a factor involves the fewest number of folds, that is, $h(a)$.*

From Theorem 3.1 we may immediately deduce a further remarkable result which we invite our readers to put to the test.

Theorem 0.4. *If for some $t \geq 2$, $\frac{t^{xy}-1}{t^x-1}$ is a factor of $\frac{t^{x'y'}-1}{t^{x'}-1}$, then xy is a factor of $x'y'$.*

The reader will also notice that, while Theorem 3.3 has some bearing on paper-folding, it is really a purely number-theoretical result.

To test Theorem 3.3, you might take $t = 2$, look at Table 1, and show first by examples that $\frac{2^{xy}-1}{2^x-1}$ is never a factor of $\frac{2^{x'y'}-1}{2^{x'}-1}$ if xy is not a factor of $x'y'$. Then find some examples where $xy | x'y'$ and $\frac{2^{xy}-1}{2^x-1} \Big| \frac{2^{x'y'}-1}{2^{x'}-1}$ (the interesting examples would not have $x = x'$). The ambitious reader might like to look at values of t different from 2, first producing a table similar to Table 1 for a particular choice of t (say, $t = 3$).

Reference

[1] Hilton, Peter, Derek Holton and Jean Pedersen, Mathematical *Reflections — in a Room with Many Mirrors*, Springer-New York, 1997.