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FOLDING REGULAR POLYGONS AND HOW IT LEADS TO A THEOREM ABOUT NUMBERS

by

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Introduction

In our paper [1] we showed how to fold a regular 7-gon — and much else besides! We showed which convex polygons could be folded by a period-2 folding procedure — these turned out to be those polygons whose number of sides, s, had the form

$$s = \frac{2^{m+n}-1}{2^n-1} \qquad (1)$$

and the procedure is then the (m, n)-folding procedure. However, 1 is not, in general, an integer; indeed the condition for it to be an integer is precisely that $n|m$. What happens[3] if $n \not| m$? Then s is some reduced fraction $\frac{b}{a}$ and the procedure described in [1] based on the (m, n)-folding procedure and the FAT algorithm produces what we call the regular star **$\{\frac{b}{a}\}$-gon**, that is, a connected sequence of edges that visits every a^{th} vertex of a regular convex b-gon (see Figure 1). We may then regard the regular N-gon as a regular star $\{\frac{N}{1}\}$-gon.

It turns out that we can only fold a regular star $\{\frac{b}{a}\}$-gon by a period-2 procedure if we can fold a regular b-gon by a period-2 procedure — this follows from the result quoted in [1] that the gcd of $t^A - 1$ and $t^B - 1$ is $t^D - 1$ where $D = \gcd(A, B)$. Thus the question remains of how to fold a regular star $\{\frac{b}{a}\}$-gon if b is not a folding number. We will answer

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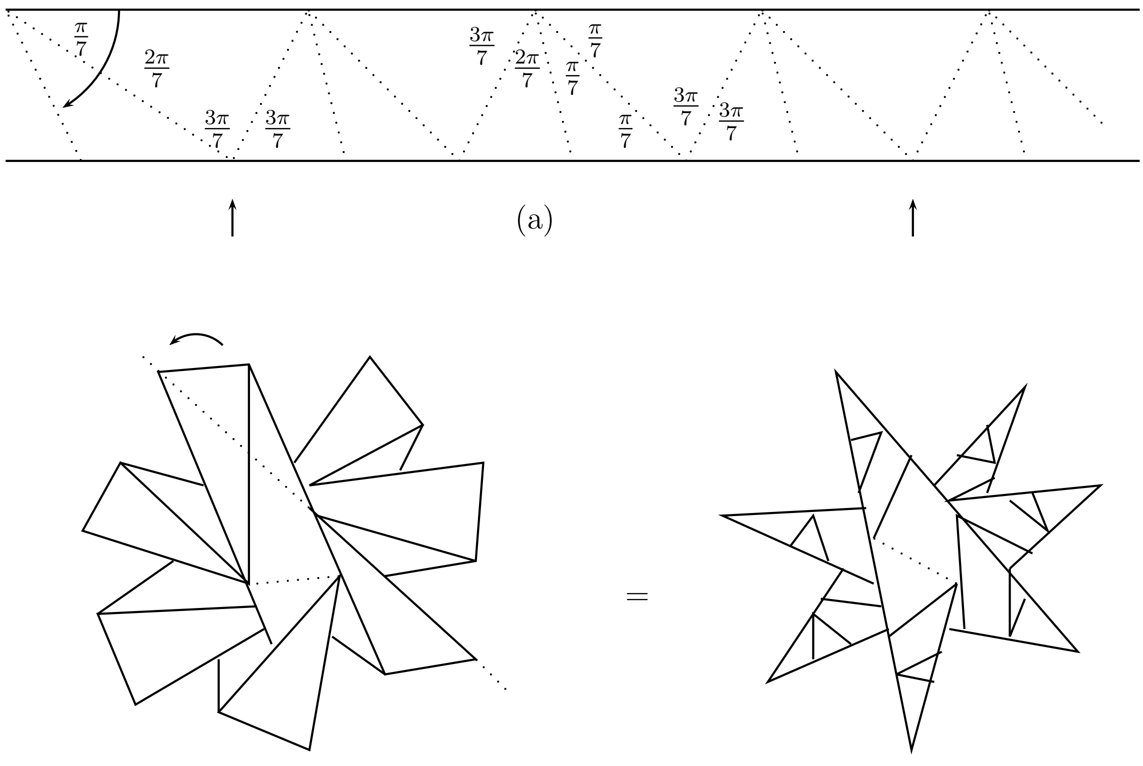
[3]Read $n \not| m$ as n does not divide m.

this question in Section 2 *if a is odd.* If a is even, there is an additional secondary procedure required; for details see [2]. (Of course, if $\frac{b}{a}$ is the reduced form of 1 then b and a are both odd.) But note that, in any case, we may always assume $a < \frac{b}{2}$, since a star $\{\frac{b}{b-a}\}$-gon is just a star $\{\frac{b}{a}\}$-gon described backwards!



(b) (c)

By executing the Fat algorithm on the *bottom* edge of the $(2,1)$-tape instead of the top, and at *every other vertex* (as indicated by the arrows in Figure 1(a)), you will obtain Figure 1(b). In Figure 1(b) the top edge of the tape visits every third vertex of a regular 7-gon. By folding back the excess around each point, as shown by the arrow in Figure 1(b), you can achieve the $\{\frac{7}{3}\}$-gon shown in Figure 1(c).

Figure 1

We will describe in Section 3 a beautiful theorem of number theory which emerges immediately from the description we give in Section 2 of the general folding procedure. Indeed, what is especially pleasing — and highly unusual — is that *precisely the same data* lead, on the one hand, to a set of explicit folding instructions for folding certain star polygons and, on the other hand, to an astonishing result in quite a different part of mathematics, namely, an algorithm for calculating what is called in number theory the *quasi-order* mod 2 of an arbitrary odd integer..

The General Folding Procedure

We will explain this procedure by a careful discussion of a *particular but not special case.* Suppose we want to construct a regular star $\{\frac{11}{3}\}$-gon, so that $b = 11, a = 3$. Writing 11 in base 2, we have[4]

$$11 \overset{(2)}{=} 1011,$$

so we see that 11 is not a folding number. Thus we know that no period-2 folding procedure could produce the regular $\{\frac{11}{3}\}$-gon. What should we do?

We proceed as we did when we wished to construct the regular convex 7-gon in [1]— we adopt our **optimistic strategy** (which means that we *assume* we've got what we want and, as we will show, we then actually *get* an arbitrarily good approximation to what we want!) Thus we assume we can fold the desired putative angle of $\frac{3\pi}{11}$ at A_0 (see Figure 2(a)) and we adhere to the same principles that we used in constructing the regular 7-gon, namely, we adopt the following rules:

1. Each new crease line goes in the forward (left to right) direction along the strip of paper.

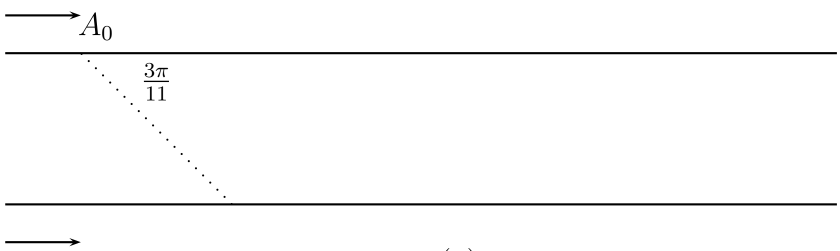
2. Each new crease line always bisects the angle between the last crease line and the edge of the tape from which it emanates.

3. The bisection of angles at any vertex continues until a crease line produces an angle of the form $\frac{a'\pi}{b}$ where a′ is an odd number; then the folding stops at that vertex and

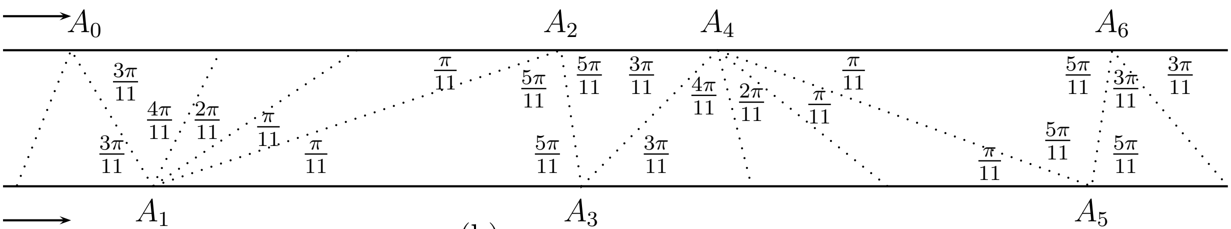
[4]Here, as in our previous papers, we use $\overset{(2)}{=}$ to mean that the number on the left, which is expressed in base ten, is equal to the number on the right, which is expressed in base two. In this case, 1011 means $1 + 2 + 8$, that is, 11.

commences at the intersection point of the last crease line with the opposite edge of the tape.

Once again the **optimistic strategy** works; and our procedure results in creased tape whose angles converge to those shown in Figure 2(b). We could denote this folding procedure as $D^1U^3D^1U^1D^3U^1$, interpreted in the obvious way on the tape — that is, the first exponent "1" refers to the one bisection (producing a line in a downward direction) at the vertices A_{6n} (for $n = 0, 1, 2, \cdots$) on the top of the tape; similarly, the "3" refers to the 3 bisections (producing creases in an upward direction) made at the bottom of the tape through the vertices A_{6n+1}; etc. However, since the folding procedure is *duplicated* halfway through, we can abbreviate the notation and write simply $\{1, 3, 1\}$, with the understanding that we alternately fold from the top and bottom of the tape as described, with the *number* of bisections at each vertex running, in order, through the values $1, 3, 1, \cdots$. We call this a **primary folding procedure of period 3** or a **period-3 folding**.



(a)

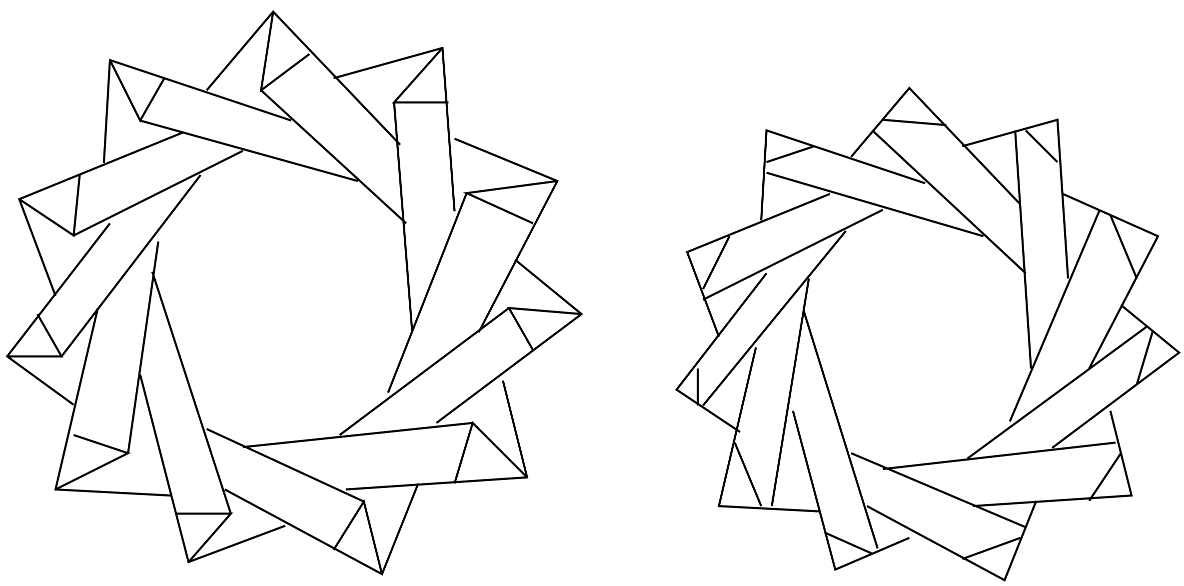


(b)

The $(1, 3, 1)$-tape for folding a $\left\{\frac{11}{3}\right\}$-gon

Figure 2

A proof of convergence for the general folding procedure of arbitrary period may be given that is similar to the one we gave for folding the regular 7-gon in [1]. We leave the details of the proof to the reader, and explore here what we can do with this $(1,3,1)$-tape. First, note that, starting with the putative angle $\frac{3\pi}{11}$ at the top of the tape, we produce a putative angle of $\frac{\pi}{11}$ at the *bottom* of the tape, then a putative angle of $\frac{5\pi}{11}$ at the top of the tape, then a putative angle of $\frac{3\pi}{11}$ at the bottom of the tape, and so on. Hence we see that we could use this tape to fold a star $\left\{\frac{11}{3}\right\}$-gon, a convex 11-gon, and a star $\left\{\frac{11}{5}\right\}$-gon. More still is true; for, as we see, if there are crease lines enabling us to fold a star $\left\{\frac{11}{a}\right\}$-gon, there will be crease lines enabling us to fold star $\left\{\frac{11}{2^k a}\right\}$-gons, where $k \geq 0$ takes any value such that $2^{k+1}a < 11$. These features, described here for $b = 11$, would be found with any odd number b. However, this tape has a special symmetry as a consequence of its *odd* period; namely, if it is "flipped" about the horizontal line half way between its parallel edges, the result is a *translate* of the original tape. As a practical matter this special symmetry of the tape means that we can use either the top edge or the bottom edge of the tape to construct our polygons. On tapes with an *even* period the top edge and the bottom edge of the tape are not translates of each other (under the horizontal flip), which simply means that care must be taken in choosing the edge of the tape used to construct a specific polygon. Figures 3(a), 3(b) show the completed $\left\{\frac{11}{3}\right\}$-, $\left\{\frac{11}{4}\right\}$-gons, respectively.



(a) The FAT $\{\frac{11}{13}\}$-gon

(b) The FAT $\{\frac{11}{4}\}$-gon

Figure 3

Now, to set the scene for the number theory of Section 3, let us look at the patterns in the arithmetic of the computations when $a = 3$ and $b = 11$. Referring to Figure 2(b) we observe that[5]

the smallest angle to the right of A_n where	is of the form $\frac{a}{11}\pi$ where	and the number of bisections at the next vertex is k where
$n = 0$	$a = 3$	$k = 3$
1	1	1
2	5	1
3	3	3
4	1	1
5	5	1

We could write this in shorthand form as follows:

$$(b=)11\begin{vmatrix}(a=)3 & 1 & 5\\(k=)3 & 1 & 1\end{vmatrix} \tag{2}$$

Observe that, had we started with the putative angle of $\frac{\pi}{11}$, then the *symbol* (2) would have taken the form

$$(b=)11\begin{vmatrix}(a=)1 & 5 & 3\\(k=)1 & 1 & 3\end{vmatrix} \tag{3}$$

In this case we wouldn't have had to go through all that arithmetic again — we would simply carry out a cyclic permutation of the columns of (2). However, it is now useful to look at the more general form taken by (2) and (3). In general we have a symbol

$$b\begin{vmatrix}a_1 & a_2 & \cdots & a_r\\k_1 & k_2 & \cdots & k_r\end{vmatrix} \tag{4}$$

where b is odd, each a_i is odd, relatively prime to b, and less than $\frac{b}{2}$; and

$$b - a_i = 2^{k_i} a_{i+1} \tag{5}$$

[5]Referring to Figure 2(b), notice that, to obtain an angle of $\frac{3\pi}{11}$ at $A_0, A_6, A_{12}, \cdots$,the folding instructions would more precisely be $U^3D^1U^1D^3U^1D^1\cdots$. But we don't have to worry about this distinction.

Moreover, there are no repeats of the a_i's (so that $a_{r+1} = a_1$).

We call r the **period** (of the paper-folding instructions) and, for convenience in the number theory we are about to explain in Section 3, we set

$$K = k_1 + k_2 + \cdots + k_r \tag{6}$$

We will now describe the arithmetical procedure for obtaining (3) without actually referring to the tape. Start with $b = 11$ and $a_1 = 1$ (this will, in fact, uniquely determine the completed symbol) and write

$$\begin{array}{c|l} 11 & \\ & 1 \\ & \end{array}$$

Now we compute: $11 - 1 = 10$, $\frac{10}{2} = 5$ (and STOP, because 5 is *odd*), and we observe that this tells us that, in this instance, (5) takes the form $11 - 1 = 2^1 5$, so we record k_1, which is 1, and a_2, which is 5, to get

$$\begin{array}{c|ll} 11 & & \\ & 1 & 5 \\ & 1 & \end{array}$$

Again we compute: $11 - 5 = 6$, $\frac{6}{2} = 3$ (and STOP, because 3 is *odd*), so that, in this instance, (5) takes the form $11 - 5 = 2^1 3$, so we record k_2, which is 1, and a_3, which is 3, to get

$$\begin{array}{c|lll} 11 & & & \\ & 1 & 5 & 3 \\ & 1 & 1 & \end{array}$$

Repeating the process, we compute $11 - 3 = 8$, $\frac{8}{2} = 4$, $\frac{4}{2} = 2$, $\frac{2}{2} = 1$ (and STOP, because 1 is *odd*), so that, in this instance, (5) takes the form $11 - 3 = 2^3 1$, so we record k_3, which is 3; and, since a_4, which is 1, is the same as a_1 we STOP without recording a_4, and draw the last vertical line to indicate that the symbol is now finished. Notice that the constructed symbol is precisely

$$\begin{array}{c|lll|} 11 & & & \\ & 1 & 5 & 3 \\ & 1 & 1 & 3 \end{array} \tag{7}$$

which is, of course, just (3).

The numbers in the bottom row of (7), when attached as superscripts to the sequence $DUDUDU...$, tell us precisely how to crease tape which can be used to fold the regular 11-gon (and, in fact, the regular $\{\frac{11}{2}\}$- and $\{\frac{11}{4}\}$-gons). Furthermore, we can see that tape with the same crease lines can also be used to fold regular star $\{\frac{11}{3}\}$- and $\{\frac{11}{5}\}$-gons.

Thus we regard (4) as encoding the general folding procedure to which we have referred. The symbol (4) tells us exactly how to fold a regular star $\{\frac{b}{a_i}\}$-gon for $i = 1, 2, \cdots, r$.

Notice that there may be star $\{\frac{b}{a}\}$-gons, with a odd, not included among the $\{\frac{b}{a_i}\}$-gons above. For example, we have the symbol

$$^{17}\left|\begin{array}{c} 1 \\ 4 \end{array}\right|$$

telling us how to fold a regular convex 17-gon (by the D^4U^4-procedure); but this does not tell us how to fold the regular star $\{\frac{17}{3}\}$-gon. For that information we require the symbol obtained by using $b = 17$ and $a_1 = 3$. Constructing this symbol, as above, we obtain

$$^{17}\left|\begin{array}{ccc} 3 & 7 & 5 \\ 1 & 1 & 2 \end{array}\right|$$

This completes the information needed to fold any $\{\frac{17}{a}\}$-gon with a odd, less than $\frac{17}{2}$, and prime to 17. We write

$$^{17}\left|\begin{array}{c|ccc} 1 & 3 & 7 & 5 \\ 4 & 1 & 1 & 2 \end{array}\right|$$

and call this the **complete symbol** for $b = 17$. Before reading the next section you might like to write down the complete symbols for $b = 43, 51, 85$. Look for patterns! (The complete symbols are given at the end of our article. Remember that, in symbol (4), b and a_i must be relatively prime.)

The Quasi-order Theorem

We now have a bonus! The information in the symbol (2) (or (3)) actually tells us the smallest number K such that either $2^K + 1$ or $2^K - 1$ will be exactly divisible by 11. In fact, in our particular example, we see, from (6), that $K = 5$ and the symbol tells us, since $r = 3$, that $2^5 - (-1)^3$, that is, $2^5 + 1$, is exactly divisible by 11 — and that for no power ℓ of 2 less than the fifth can either $2^\ell + 1$ or $2^\ell - 1$ be divisible by 11. This is because for the entries in the symbol (4), generated as described, for given b and any suitable a_1, it is always the case that

$$2^K - (-1)^r \text{ is exactly divisible by } b;$$

and that there is no smaller power ℓ of 2 such that $2^\ell + 1$ or $2^\ell - 1$ is divisible by b. We call K the **quasi-order of 2 mod *b*** and refer to the result as the **Quasi-order Theorem**.

The Quasi-order Theorem explains the patterns that you may have found in the complete symbols for $43, 51, 85$. Of course, to deduce the quasi-order of 2 mod b for any odd number b, it suffices to extract any symbol (i.e., with a given a_1) from the complete symbol. What are the quasi-orders of 2 mod 43, of 2 mod 51, of 2 mod 85? You will find the answers with the complete symbols at the end of our article.

Here is a particularly interesting example of the Quasi-order Theorem — we'll explain why. Choose $b = 641$, and $a_1 = 1$ and construct the symbol. Try constructing the symbol for yourself before you look carefully at it, to give you some practice with the algorithm involving the repeated use of (5).

$$641\left|\begin{matrix} 1 & 5 & 159 & 241 & 25 & 77 & 141 & 125 & 129 \\ 7 & 2 & 1 & 4 & 3 & 2 & 2 & 2 & 9 \end{matrix}\right|$$

We can now calculate that $K = 7 + 2 + 1 + 4 + 3 + 2 + 2 + 2 + 9 = 32$, and observe that $r = 9$, so that the Quasi-order Theorem tells us that

$$2^{32} - (-1)^9 = 2^{32} + 1 \text{ is exactly divisible by } 641!$$

We have just *proved* that the fifth Fermat number $2^{2^5}+1$ is *not prime*. This fact was originally discovered by Leonhard Euler, and was the first demonstration that not all Fermat numbers (see [1]) are prime.

If you feel you are now ready for a proof of the Quasi-order Theorem and for further ideas in the same direction, along with some interesting questions that you could think about, you should consult the references [2, 3].

Some Complete Symbols

43	1	21	11	3	5	19	7	9	17	13	15
	1	1	5	3	1	3	2	1	1	1	2

51	1	25	13	19	5	23	7	11
	1	1	1	5	1	2	2	3

85	1	21	3	41	11	37	7	39	23	31	27	29	9	19	33	13
	2	6	1	2	1	4	1	1	1	1	1	3	2	1	2	3

The quasi-order of 2 mod 43 is 7; the quasi-order of 2 mod 51 is 8; the quasi-order of 2 mod 85 is 8. In fact, $43 \mid 2^7 + 1$, $51 \mid 2^8 - 1$, $85 \mid 2^8 - 1$.

References

[1] Hilton, Peter and Jean Pedersen, Constructing Regular 7-gons — and Much Else Besides, **Parabola 34** No 1 (1998) 2–12.

[2] Hilton, Peter and Jean Pedersen, Geometry in Practice and Numbers in Theory, **Monographs in Undergraduate Mathematics 16** (1987), 37 pp. (Available from the Department of Mathematics, Guilford College, Greensboro, NC, 27410.)

[3] Hilton, Peter and Jean Pedersen, On factoring $2^k \pm 1$, **The Mathematics Educator 5** No. 1 (1994), 29–32.