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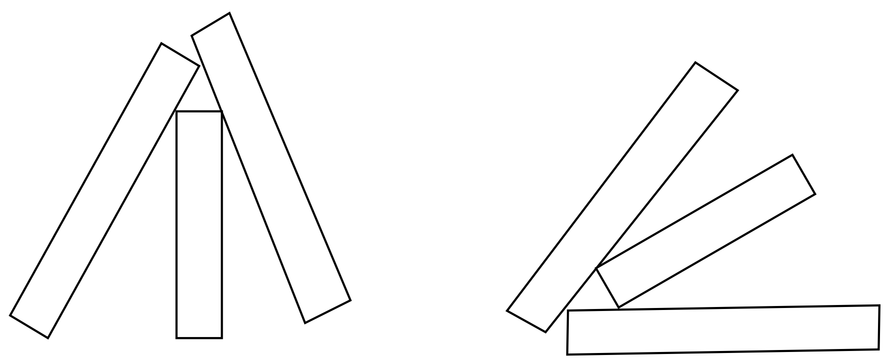
SOLUTIONS TO PROBLEMS 1016, 1025-1034

Q. 1016 Show how six cylindrical pencils of equal radius each with neither end sharpened can be put into mutual contact along their curved surfaces.

ANS. We can put three (equal) cylindrical pencils into mutual contact while they lie on a table as shown in either diagram below.



Here each diagram represents three pencils on the table. We can now lift either configuration and put it down on top of the other to get six (equal) cylindrical pencils in mutual contact. [It is harder, but still possible, to have seven pencils in mutual contact. This is left open.]

Q.1025 Find the smallest number that when divided by 29 leaves the remainder 23 and that when divided by 37 leaves the remainder 31.

ANS. We first have to solve $29x + 23 = 37y + 31$, that is $29x = 37y + 8$, in integers. Positivity will be handled later. The Euclidean algorithm for 29 and 37 is:

$$37 = 29 + 8,\ 29 = 3 \times 8 + 5,\ 8 = 5 + 3,\ 5 = 3 + 2,\ 3 = 2 + 1.$$

We reverse this chain of equations to obtain:

$$1 = 3 - 2 = 2 \times 3 - 5 = 2 \times 8 - 3 \times 5 = 11 \times 8 - 3 \times 29 = 11 \times 37 - 14 \times 29$$

and thus $8 = 88 \times 37 - 112 \times 29$. The equation $29x = 37y + 8$ can then be written as:

$$29x = 37y + 88 \times 37 - 112 \times 29,$$

that is

$$29 \times (x + 112) = 37 \times (y + 88).$$

As 29 and 37 have no common factor, there must be some integer t such that $(29 \times (x+112) = 29 \times 37t$, that is $x + 112 = 37t$, or $x = 37t - 112$. Thus $29x + 23 = 29 \times 37t - 29 \times 112 + 23 = 1073t - 3225$. So we need $t > \frac{3225}{1073}$ and thus $t = 4, 29x + 23 = 1067$.

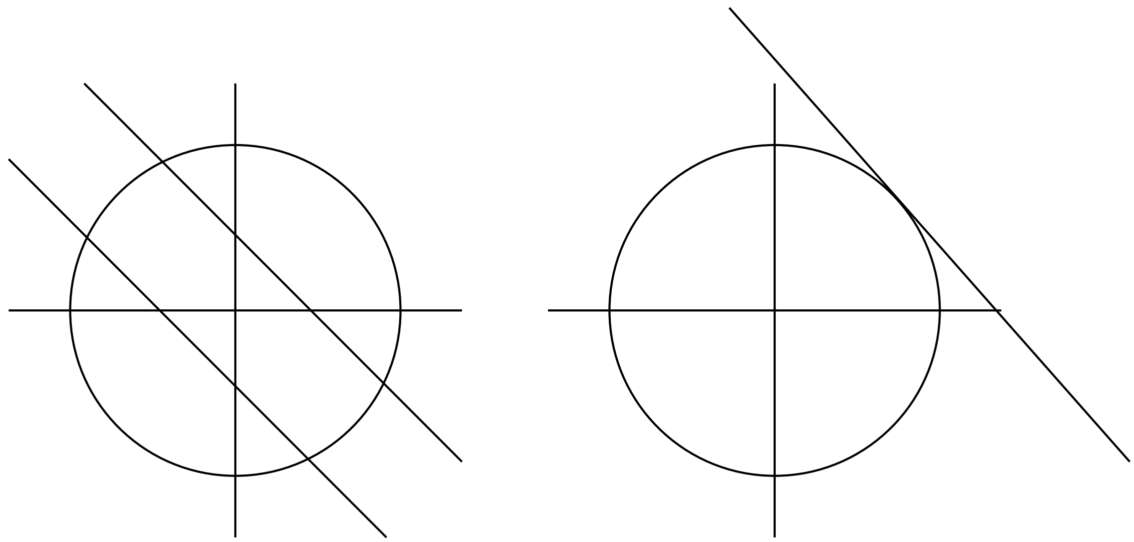
Q.1026 Find the smallest number that when divided by 29 leaves the remainder 23, that when divided by 37 leaves the remainder 31 and that when divided by 43 leaves the remainder 41.

ANS. In view of the calculations immediately above, we have to solve $1073t - 3225 = 43z + 41$ in integers and then seek the least such 3 with $433 + 41 > 0$. This is done by the same method as before. So we have to re-write the equation as $1073 \times (t - 68586) = 43 \times (z - 1711384)$. It turns out that the answer is 43987.

Q.1027 Find the maximum of $3x + 4y$ where x, y are subject to $x^2 + y^2 = 1$.
Interpret your answer geometrically.
Now find the maximum of $8x + 4y + 1$ where x, y, z are subject to $x^2 + y^2 + z^2 = 1$.

ANS. (a) We sketch the circle $x^2 + y^2 = 1$ and various lines $3x + 4y = c$, for c constant, in the diagram on the left. The greatest value of c for which the line intersects the circle is where the line is tangent as shown in the diagram on the right. Calculations shows that the point of tangency is $(3/5, 4/5)$ and there the relevant value of c is $3 \times 3/5 + 4 \times 4/5 = 5$. Alternatively, we may let $x = \cos\theta,\ y = \sin\theta$ in parametrising the circle and then use calculus to find the maximum of $3\cos\theta + 4\sin\theta$. This yields $-3\sin\theta + 4\cos\theta = 0,\ \tan\theta = 4/3$ and the same solution is obtained.

(b) This problem is essentially the three -dimensional analogue of the previous problem. We seek the point (ξ, η, ζ) on the sphere such that the tangent is parallel to the plane $8x+4y+1 = c$; this must be where $\xi/8 = \eta/4,\ \zeta = 0$. Hence $\xi = \pm 2/\sqrt{5},\ \eta = \pm 1/\sqrt{5},\ \zeta = 0$. Clearly the '+' in the $\pm$ is appropriate for a maximum, which thus must be $8 \times 2/\sqrt{5} + 4 \times 1/\sqrt{5} + 1$.



Q.1028 If $\binom{n}{k}$ denotes the binomial co-efficient, show that $\sum_{k=0}^{n} k\binom{n}{k} = 2^{n-1}$.

ANS. The binomial expansion for $(1+x)^n$ is

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

Differentiate this with respect to x to obtain:

$$n \cdot x^{n-1} = \binom{n}{1} + 2 \cdot \binom{n}{2}x + \cdots + n\binom{n}{n}x^{n-1}.$$

We put $x = 1$ in this to obtain the correct version of the required formula

$$n \cdot 2^{n-1} = \binom{n}{1} + 2 \cdot \binom{n}{2}2 + n \cdot \binom{n}{n}2^{n-1}.$$

Q.1029 Show that $\sqrt[3]{7+5\sqrt{2}} + \sqrt[3]{7-5\sqrt{2}} = 2$.

ANS. Note that $(1+\sqrt{2})^3 = 1+3\sqrt{2}+6+2\sqrt{2} = 7+5\sqrt{2}$, and likewise $(1-\sqrt{2})^3 = 7-5\sqrt{2}$. The required equality follows.

Q.1030 A citizen is sitting in a rowing boat floating in a small lake. There is a large steel anchor in the boat. If the citizen throws the anchor overboard does the level of the lake rise or fall? Does the level of the boat relative to the shore rise or fall?

ANS. When the anchor is in the boat, it is adding its weight to the weight of the boat and the citizen and so displaces its own weight in water. When it is at the bottom of the lake it displaces only its own volume in water, and this is a lesser quantity of water displaced. Hence the level of the lake falls when the anchor is thrown overboard.

Q.1031 Three circles each of radius a are in mutual contact. Find the radius of the circle that circumscribes all three.

ANS. We will use co-ordinate geometry to save having to think. The centres of the circles form an equilateral triangle in the plane. We may take $(a,0),(-a,0)$ and $(0,\sqrt{3}a)$ to be the centres of these circles. Then $(0,a/\sqrt{3})$ is the centre of the equilateral triangle and its distance to $(0,(\sqrt{3}+1)a)$, the furthest point of the circle centred at $(0,\sqrt{3}a)$ and radius a is $(\sqrt{3}+1-1/\sqrt{3})a$. This is the radius of the circumscribing circle.

Q.1032 Four spheres each of radius a are in mutual contact. Find the radius of the sphere that circumscribes all four.

ANS. This problem can also be turned into an exercise in co-ordinate geometry. We take the centres of the four spheres to be

$$(b,b,b),\ (b,-b,-b),\ (-b,-b,b),\ (-b,b,-b).$$

These are seen to be the vertices of a regular tetrahedron whose vertices are $\sqrt{2}b$ apart and so we want $b = \sqrt{2}a$. The centre of the tetrahedron is at $(0,0,0)$ and is distant $\sqrt{6}a$ from $(\sqrt{2}a,\sqrt{2}a,\sqrt{2}a)$. So the furthest point from the origin on the sphere centred at $(\sqrt{2}a,\sqrt{2}a,\sqrt{2}a)$ is $(1+\sqrt{6})a$.

Q.1033 What rate of interest paid in advance on a 1-year loan is equivalent to an interest rate of $r\%$ paid at the end of the term of the loan?

ANS. If we borrow \$100 for a year paying $\$r'$ interest in advance, we are in effect paying $\$r'$ in interest at the end of the year for borrowing \$ $(100-r')$. Thus this is equivalent to $r = r'/(100-r')$ per cent interest paid at the end of the year. This equation determines r' in terms of r.

Q.1034 Suppose that the diagram shown has been drawn in pencil. It is desired to draw ink squares over the pencil lines so as to cover the entire design with ink. What is the minimal number of ink squares needed to carry out this task?

ANS. We begin thinking about this rather unusual problem by considering certain simplifications of it.

If instead of the original diagram we have a square diagram with 9 points and 3 lines in each direction, the most efficient method of colouring in the diagram by squares is to draw one 2×2 square to cover the perimeter and then two 1×1 squares to cover the other lines.

If instead of the original diagram we have a square diagram with 16 points and 4 lines in each direction, the most efficient method of colouring in the diagram by squares is to draw four 2×2 squares each of which has one corner in common with the original diagram.

If instead of the original diagram we have a square diagram with 25 points and 5 lines in each direction, the most efficient method of colouring in the diagram by squares is to draw four 3×3 squares each of which has one corner in common with the original diagram and then draw two 2×2 squares to finish the task.

The above smaller tasks should now have produced some of the techniques needed for the original problem in which we have 36 points and 6 lines in each direction. Choose two opposite corners of the original diagram and make each a corner of a coloured 3×3 square and also of a coloured 2×2 square. Make each of the other corners of the original diagram the corner of a coloured 4×4 square and also of a coloured 1×1 square.