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SOLUTIONS TO PROBLEMS 1035-1042

Q. 1035 Find all positive integers n and m such that n is a factor of $4m-1$ and m is a factor of $4n-1$.

ANS. We write $n|(4m-1)$ and $m|(4n-1)$ to mean that n is a factor of $4m-1$ and m is a factor of $4n-1$, and we may suppose that $m \leq n$. Then $4m-1 < 4m \leq 4n$ and so if $4m-1 = kn$ we have $k = 1, 2$ or 3. Now $k \neq 2$ since $4m-1$ is odd.

If $k=1$, then m divides $4(4m-1)-1 = (16m-5)$ so $m = 1$ or 5.
This gives the solutions $(m,n) = (1,3), (5,19)$.

If $k=3$, we have $m|4n-1$ and so $3m|(12n-3) = (16m-7)$. Thus $m|7$, giving the solutions $(m,n) = (1,1), (7,9)$.

Q. 1036 Prove (without using induction) that $2^{4n} - 15n - 1$ is divisible by 225, for all positive integers n.

ANS.

$$\begin{aligned} 2^{4n} - 15n - 1 &= 16^n - 15n - 1 = (15+1)^n - 15n - 1 \\ &= 15^n + ... +^n C_2 15^2 + 15n + 1 - 15n - 1 \\ &= 15^n + ... +^n C_2 15^2 \end{aligned}$$

which is clearly divisible by $15^2 = 225$.

Q. 1037 Suppose n is a positive integer and let $f(x) = \frac{x}{1+n^2x^2}$. Without using calculus, show that $f(x) \leq \frac{1}{2n}$. (Hint: Look at $\frac{1}{f(x)}$.)

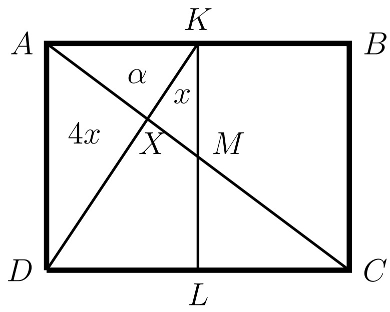
ANS. We use the well-known fact that the arithmetic mean $\frac{1}{2}(a+b)$ of two numbers a, b is at least as big as their geometric mean $\sqrt{ab}$. Then

$$\frac{1}{f(x)} = \frac{1}{x} + n^2 x \geq 2\sqrt{n^2} = 2n$$

so $f(x) \leq \frac{1}{2n}$.

Q. 1038 Let $ABCD$ be a rectangle, and K,L be the midpoints of AB and CD respectively. Suppose AC and KD meet at X. Find the ratio of the area of ΔKXA to the rectangle $ABCD$.

ANS.



Let area of $\Delta AXK = \alpha$, and area of $\Delta KXM = x$. Now ΔADX is similar to ΔKMX and $|KM| = \frac{1}{2}|AD|$. So

$$\text{area of } \Delta ADX = 4 \times \text{area of } \Delta KMX = 4x.$$

Similarly,

$$\text{area of } \Delta ABC = 4 \times \text{area of } \Delta AKM = 4(x+\alpha).$$

and

$$\text{area of } AKLD = 2 \times \text{ area of } \Delta AKD = 2(4x+\alpha).$$

Since these are both half the rectangle $ABCD$, we can write $4x + \alpha = 2(x+\alpha)$ giving $2x = \alpha$. Now $\alpha + 4x = \frac{1}{4}$ area of $ABCD$ and so $\alpha = \frac{1}{12}$ area of $ABCD$.

Q. 1039 Let M, N be positive integers. If $x^M(1-x)^N$ is divided by $1+x^2$ giving a remainder of $ax+b$ show that $a = (\sqrt{2})^N \sin\Big((2M-N)\frac{\pi}{4})\Big)$ and $b = (\sqrt{2})^N \cos\Big((2M-N)\frac{\pi}{4})\Big)$.
(Hint: Complex numbers may be useful.)

ANS. Write

$$x^M(1-x)^N = p(x)(1+x^2) + ax + b.$$

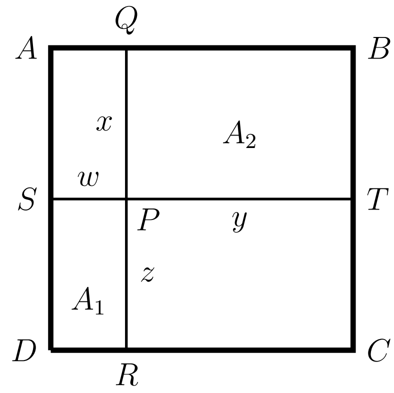
Substituting $x = i$ in this equation, $i^M(1-i)^N = ai + b$. We can write $i = e^{i\frac{\pi}{2}} = \text{cis}(\frac{\pi}{2})$ and $(1-i) = \sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}\text{cis}(\frac{\pi}{4})$ and substitute to obtain

$$e^{i\frac{\pi}{4}(2M-N)}(\sqrt{2})^N = ai + b \quad \text{or } (\sqrt{2})^N\text{cis}(\frac{\pi}{4}(2M-N)) = a + ib.$$

Now equate real and imaginary parts.

Q. 1040 Let $ABCD$ be a unit square and let ST be any line passing through the square which is parallel to AB and QR be a line passing through the square which is parallel to the side BC. Suppose that these lines meet at P. Show that if the rectangle $QPTB$ has area larger than $\frac{1}{4}$ then the rectangle $DSPR$ has area smaller than $\frac{1}{4}$.

ANS.



Write $|QP| = x = \frac{1}{2} - \epsilon$, $|PT| = y = \frac{1}{2} + \epsilon$, $|SP| = w = \frac{1}{2} - \delta$, $|PR| = z = \frac{1}{2} + \delta$.
Then $A_1 A_2 = (\frac{1}{4} - \epsilon^2)(\frac{1}{4} + \epsilon^2) < \frac{1}{16}$. Hence if $A_2 > \frac{1}{4}$ we have $A_1 < \frac{1}{4}$.

Q. 1041 Suppose we have $n+1$ positive integers all less than or equal to $2n$. Prove that among this list there must be an integer that divides one of the other integers.

ANS. Write the $n+1$ integers as $a_1, a_2, ..., a_{n+1}$. Each of these numbers can be written as a power of 2 times an odd number, i.e.

$$a_i = 2^{\alpha_i} q_i, \quad \text{where} \quad q_i \quad \text{is odd.}$$

Now since each number is less than $2n$, each of the $n+1$ q_i's is less than $2n$. But there are only n odd numbers between 1 and $2n$ and so two of the q_i's must be equal. Call this common value Q, then we have two of the a_i's of the form

$$a_s = 2^{\alpha_s} Q \quad \text{and} \quad a_t = 2^{\alpha_t} Q = 2^{\alpha_t - \alpha_s} a_s$$

with $\alpha_s \le \alpha_t$ and so a_s is a factor of a_t.

Q. 1042 The famous sleuth Hercule Poirot has discovered the following facts.

- Professor Edelstein was not in the study at the time of the murder.
- Either Lady Adeline committed the murder or Sir Benjamin did.
- If Countess Delilah was not in the study then Professor Edelstein was in the study.
- If Sir Benjamin committed the murder then Cecil was in the dining room at the time of the murder and Countess Delilah was not in the study.

Who committed the murder?

ANS. Suppose that Sir Benjamin committed the murder.
Then Countess Delilah was not in the study.
But this means that Professor Edelstein was in the study, which contradicts the facts.
Hence Sir Benjamin is innocent, and Lady Adeline was the murderess.