SOLUTIONS TO PROBLEMS 1051–1056

Q.1051 What is the fractional derivative

$$\frac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$$

of $f(x) = 1/\sqrt{x}$ (see the article on fractional calculus in Parabola, Vol. 35, No. 2)? **ANS.**

$$\frac{d^{1/2}x^{-\frac{1}{2}}}{dx^{1/2}} = \frac{d}{dx}\left(\frac{d^{-1/2}x^{-\frac{1}{2}}}{dx^{-1/2}}\right)$$

where

$$\begin{aligned}\frac{d^{-1/2}x^{-\frac{1}{2}}}{dx^{-1/2}} &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^x \frac{y^{-\frac{1}{2}}}{(x-y)^{-\frac{1}{2}+1}}\,dy \\ &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^x \frac{1}{\sqrt{xy-y^2}}\,dy.\end{aligned}$$

To simplify the above integral first complete the square in the denominator, then

$$\frac{d^{-1/2}x^{-\frac{1}{2}}}{dx^{-1/2}} = \frac{1}{\Gamma(\frac{1}{2})}\int_0^x \frac{1}{\sqrt{\frac{x^2}{4}-(y-\frac{x}{2})^2}}\,dy.$$

Now consider the change of variables from y to θ via

$$\begin{aligned}y &= \frac{x}{2}+\frac{x}{2}\sin\theta \\ dy &= \frac{x}{2}\cos\theta d\theta \\ y=0 &\Rightarrow \theta=-\pi/2 \\ y=x &\Rightarrow \theta=\pi/2.\end{aligned}$$

With this change of variables we can write

$$\frac{d^{-1/2}x^{-\frac{1}{2}}}{dx^{-1/2}} = \frac{1}{\Gamma(\frac{1}{2})}\int_{-\pi/2}^{\pi/2} \frac{\frac{x}{2}\cos\theta\,d\theta}{\sqrt{\frac{x^2}{4}-\frac{x^2}{4}\sin^2\theta}}.$$

To simplify this integral further use the trigonometric identity

$$\cos^2\theta+\sin^2\theta=1,$$

to cancel out the denominator. Then

$$\begin{aligned}\frac{d^{-1/2}x^{-\frac{1}{2}}}{dx^{-1/2}} &= \frac{1}{\Gamma(\frac{1}{2})}\int_{-\pi/2}^{\pi/2} d\theta \\ &= \frac{\pi}{\Gamma(\frac{1}{2})} \\ &= \sqrt{\pi}\end{aligned}$$

Finally the fractional derivative of $1/\sqrt{x}$ of order one half is

$$\begin{aligned}\frac{d^{1/2}x^{-\frac{1}{2}}}{dx^{1/2}} &= \frac{d}{dx}\left(\frac{d^{-1/2}x^{-\frac{1}{2}}}{dx^{-1/2}}\right) \\ &= \frac{d}{dx}\sqrt{\pi} \\ &= 0.\end{aligned}$$

Q.1052 In a class of thirty students what is the probability that two or more students have their birthdays on the same day?

ANS. The probability p that two or more students have their birthdays on the same day is equivalent to one minus the probability that all the students have their birthdays on a different day. To calculate the probability that all students have their birthdays on different days start with the first student who can have their birthday on any of the 365 days, then the second student can have their birthday on any of the remaining 364 days and the third student on any of the remaining 363 days and so on. Hence we have

$$\begin{aligned}p &= 1-\left(\frac{365}{365}\right)\left(\frac{364}{365}\right)\left(\frac{363}{365}\right)\cdots\left(\frac{365-(n-1)}{365}\right) \\ &= 1-\frac{365!}{(365-n)!365^n}\end{aligned}$$

In the case $n = 30$ the above product reduces to $p = .7063...$, i.e., in a class of thirty students there is a better than seventy percent probability that two or more students share the same birthday.

NOTE: It is interesting how quickly this probability grows. For example in a class of 40 students it is 89%, in a class of 50 students it is 97%, and in a class of 60 it is 99.4%.

Q.1053 A continued fraction is an expression of the form

$$[n_0, n_1, n_2, n_3, \ldots] = n_0 + \frac{1}{n_1 + \frac{1}{n_2+\frac{1}{n_3+\cdots}}}.$$

A useful feature of continued fractions is that they can be used to provide a set of integers to approximate an irrational number. As an example consider

$$\sqrt{14} = 3.7416573867739413856...$$

which can be written as the continued fraction

$$[3, 1, 2, 1, 6, 1, 2, 1, 6, 1, 2, 6, 1, \ldots].$$

What is the continued fraction for the Golden Section $\phi = (1 + \sqrt{5})/2$?

ANS.

$$\phi = \frac{1 + \sqrt{5}}{2}$$

can be written as the continued fraction

$$[1, 1, 1, 1, \ldots] = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}.$$

It is easy to verify this result but not so easy to derive it. To verify the result note that if

$$\phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}.$$

then we can write

$$\phi = 1 + \frac{1}{\phi}$$

which is equivalent to the quadratic

$$\phi^2 - \phi - 1 = 0.$$

The formula for the roots of a quadratic then gives the result

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

The result can be derived by recognizing that the continued fraction representation of a number ω can be generated by a simple iteration scheme;

$$\begin{aligned} n_j &= \lfloor \omega_j \rfloor \\ \omega_{j+1} &= \frac{1}{\omega_j - n_j} \end{aligned}$$

where $\omega_0 = \omega$ and $\lfloor \omega_j \rfloor$ indicates the nearest integer not exceeding ω_j. In the case of

$$\omega = \frac{1 + \sqrt{5}}{2} = 1.61803...$$

we have

$$\omega_0 = \frac{1 + \sqrt{5}}{2} \qquad n_0 = \lfloor 1.61803... \rfloor = 1.$$

Then

$$\begin{aligned}\omega_1 &= \frac{1}{\omega_0 - 1} = \frac{1}{\frac{1+\sqrt{5}}{2} - 1} \\ &= \frac{2}{-1+\sqrt{5}} \\ &= \frac{2(-1-\sqrt{5})}{(-1+\sqrt{5})(-1-\sqrt{5})} \\ &= \frac{-2(1+\sqrt{5})}{(1-5)} \\ &= \frac{1+\sqrt{5}}{2} \\ &= \omega_0\end{aligned}$$

Hence $\omega_j = \omega$ for all j and $n_j = n_0 = 1$ for all j.

Q. 1054 What is the value of the infinite square root

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}}}$$

ANS. Suppose the value is a. Squaring the expression, we get

$$a^2 = 1 + \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}} = 1 + a.$$

So $a = \dfrac{1 \pm \sqrt{5}}{2}$. Since $a > 0$, $a = \dfrac{1+\sqrt{5}}{2}$.

Note: This is (again) the Golden Section.

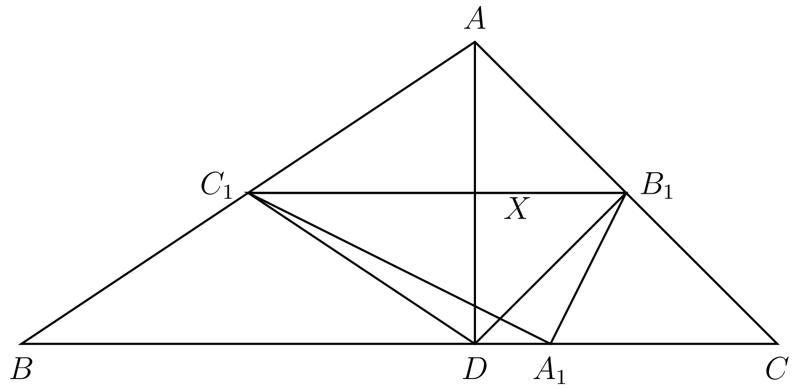
Q. 1055 Let ABC be a triangle with A_1, B_1, C_1 the midpoints of the sides BC, CA, AB respectively.

Let D be the foot of the perpendicular from A to BC.

Show that the triangles B_1C_1D and $B_1C_1A_1$ are congruent.

ANS. Let ABC be a $\triangle$ with A_1, B_1, C_1 the midfpoints of the sides BC, CA, AB respectively. Let D be the foot of the perpendicular from A to BC. Show that

$$DB_1C_1D \equiv AB_1C_1A_1$$



$$AC_1 \| B_1A_1 \qquad \text{and} \qquad A_1C_1 \| B_1A.$$

So $AC_1A_1B_1$ is a parallelogram.

Thus $\triangle C_1A_1B_1 \equiv \triangle AC_1B_1$.

Also $C_1B_1 \| BC$ and so $AXC_1 = 90^\circ$

Now in the triangles AXC_1 and XC_1D, C_1X is common and $AX = XD$ (since $AC_1 = C_1B$).

So $\triangle AXC_1 \equiv \triangle XC_1D$(SAS)

Thus $AC_1 = C_1D$

Similarly $B_1D = AB_1$

So $\triangle AC_1B_1 \equiv \triangle C_1DB_1$.

Thus $\triangle C_1DB_1 \equiv \triangle C_1A_1B_1$.

Q. 1056 Alice and Bob arrived at the dairy with their jugs, each asking for 2 litres of milk. The assistant serving them found that he had nothing to measure out the milk: all he had was an 80-litre can full of milk. Alice informed him that her jug held exactly 5 litres and Bob said that his held 4 litres.

(a) Find out how the assistant was able to measure out the 2 litres that Alice wanted into her jar.

(b) After Alice left, the assistant realised that there was another 80-litre can full of milk in the storeroom. How did he give Bob the 2 litres that he wanted?

ANS.

(a) We will use ordered triples (n_C, n_A, n_B) to indicate the amount of milk in the 80-litre can, Alice's 5-litre jug and Bob's 4-litre jug. Thus the original situation is represented by $(80, 0, 0)$ and pouring 5 litres from the (full) 80-litre can is represented by $(80, 0, 0) \rightarrow (75, 5, 0)$. The solution is then

$$\begin{array}{rcl} (80,0,0) & \rightarrow & (75,5,0) \\ & \rightarrow & (75,1,4) \\ & \rightarrow & (79,1,0) \\ & \rightarrow & (79,0,1) \\ & \rightarrow & (74,5,1) \\ & \rightarrow & (74,2,4) \\ & \rightarrow & (78,2,0) \end{array}$$

(b) This time we will use our ordered triples to indicate the amount of milk in the cans and Bob's 4-litre jug respectively. Thus (a) has left us with the situation $(78, 80, 0)$, where Alice departed with her 2 litres of milk. The solution is now

$$(78, 80, 0) \to (78, 76, 4) \to (80, 76, 2)$$

and we can put our full 80-litre can away again.