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AFTER SQUARING THE DIGITS OF A NUMBER

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In this paper we note the following property shared by all positive integers. Given any positive integer x, if we take the sum of the squares of its digits, perform the same process on the resulting number and go on repeating the process, then after a finite number of steps we arrive either at the number 4 or at the number 1. If we get 1, then the above process will always result in 1 from then on: if we get 4, then after every 8 steps the above process will result in 4 again. So, the number 1 is a "fixed point" whereas the numbers

$$4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4$$

form a "cycle" (where an arrow denotes the process mentioned above). In other words, if we start with any positive integer and go on repeating the process described above then it is going to result either in the "fixed point" 1, or in the "cycle of 4".

Examples:

1. $169 \to 118 \to 66 \to 72 \to 53 \to 34 \to 25 \to 29 \to 85 \to 89 \to 145 \to 42 \to 20 \to 4$
2. $7 \to 49 \to 97 \to 130 \to 10 \to 1.$

Now we will prove the above result after introducing some notation. Let $\mathbb{N}^+$ be the set of all positive integers. We define a function S from $\mathbb{N}^+$ into $\mathbb{N}^+$ as follows: Let $x \in \mathbb{N}^+$ and set $x = (x_1, x_2, \ldots, x_n)$ where the x_i are the digits of x: so $1 \leq x_1 \leq 9$ and $0 \leq x_i \leq 9$ for $i = 2, \ldots, n$. We define $S(x) = x_1^2 + x_2^2 + \cdots + x_n^2$.
Since the expression Σx_i^2 is symmetric in the x_i, the value of $S(x)$ does not change if we permute the digits of x. We denote by $S^k(x)$ the number $(\ldots S(S(x)\ldots)$ repeated k times.

Theorem 0.1 *For any $x \in \mathbb{N}^+$, there exists a positive integer k (depending on x) such that*

$$S^k(x) = 1 \quad \textit{or} \quad 4.$$

Moreover, $S^i(1) = 1$ and $S^{8i}(4) = 4$ for any positive integer i.

The idea behind the proof is to verify the result individually for numbers of lower digits and then use induction on the number of digits. For this we need the following little lemma :

Lemma 0.2 $10^{n-2} > 81n$ *for all* $n \geq 5$.

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Proof: We use induction on n. For $n = 5$, the result is true since

$$10^3 = 1000 > 81 \times 5 = 405.$$

Let us assume that the result is true for a certain $n \geq 5$, i.e. $10^{n-2} > 81n.$

$$\text{But} \qquad 9 \times 10^{n-2} > 9 \times 9 = 81 \quad \text{if} \quad n \geq 5.$$

Adding the above two strict inequalities we get

$$\begin{aligned} 10^{n-2} + 9 \times 10^{n-2} &> 81n + 81, \\ \text{or} \quad 10^{n-1} &> 81(n+1). \end{aligned}$$

Therefore the result is true for $n + 1$, and hence by induction the lemma is proved.

Proof of Theorem 1:

First, we verify the result individually for all positive integers having one or two digits. Since the value of $S(x)$ is independent of the permutation of digits of x, we need verify the result for the following numbers only:

1	11								
2	12	22							
3	**13**	**23**	33						
4	14	24	34	**44**					
5	15	25	35	45	55				
6	16	26	36	46	56	66			
7	17	27	37	47	57	67	77		
8	18	**28**	38	48	58	**68**	78	88	
9	**19**	29	39	**49**	59	69	**79**	89	99

By explicit calculation we find that the numbers which are printed in dark type in the above list end up at 1, and the rest end up at 4. We have verified the result for all positive integers up to 99.

Now, for any positive integers x with $100 \leq x \leq 149, S(x) < 99$: because if $100 \leq x \leq 149$, then x is of the form x_1, x_2, x_3 with $x_1 = 1$, $0 \leq x_2 \leq 4$, $0 \leq x_3 \leq 9$, and $S(x) = x_1^2 + x_2^2 + x_3^2 \leq 1^2 + 4^2 + 9^2 = 98 < 99$. So the result is verified up to 149.

Now for $150 \leq x \leq 189$, $S(x) < 149$: because if $150 \leq x \leq 189$ then x is of the form x_1, x_2, x_3 with $x_1 = 1$, $5 \leq x_2 \leq 8$, $0 \leq x_3 \leq 9$. So we have verified the result up to 189. By a similar argument we see that for $190 \leq x \leq 199$, $S(x) < 189$, and for $200 \leq x \leq 299$, $S(x) < 199$.

For any three digit number x, $S(x) \leq 9^2 + 9^2 + 9^2 = 243 < 299$: so the result is true up to 999. Also for any $x \leq 99999$, $S(x) \leq 9^2 + 9^2 + 9^2 + 9^2 + 9^2 = 405 < 999$. Therefore the result holds for all positive integers with at most 5 digits.

Now we use induction on the number of digits, as follows: first we note that $S(x) \leq 9^2 \times n = 81n$, for all positive integers x with at most n digits. It follows from the lemma that if x is a positive integer with n digits $(n \geq 5)$ and $y = S(x)$, then $y \leq 81n < 10^{n-2}$, which is the least positive integer with $(n-1)$ digits. Therefore, y has at most $(n-1)$ digits. By the induction hypothesis, the result is true for all positive integers with at most $(n-1)$ digits. So, there exists a positive integer u such that $S^u(S(x)) = S^u(y) = 1$ or 4. So $S^k(x) = 1$ or 4, where $k = u + 1$. This proves the first part of the theorem.

The second part is obvious, for $S(1) = 1$ and $S^{8i}(4) = 4$ for any positive integer i (by induction on i).

Note: The property of numbers which we have seen here holds for numbers of base 10. Now we will see what the result becomes when we change the base.

1. For base 2, we get only one "fixed point" 1.
2. For base 4, we get only one "fixed point" 1.
3. For base 5, we get "fixed points"1, 23 and 33 and a "cycle" "$4 \to 31 \to 20 \to 4$".
4. For base 8, we get "fixed points" 1, 24, 32 and 64, and two "cycles" $4 \to 20 \to 4$ and $5 \to 31 \to 12$.

The proofs of these results are based on similar arguments as numbers in base 10.

MATHEMATICAL POSTAGE STAMP

On 20th August this year, the French Post Office issued a stamp to commemorate the 400th anniversary of the birthday of Pierre de Fermat.