SOLUTIONS TO PROBLEMS 1098-1104

Q1098 Consider the 2000-digit number which begins with 1000 4's, next has 999 8's and ends in a 9. Find its square root.

ANS. The number 1000 is surely not crucial! So consider

$$a_n = \underbrace{44\ldots 4}_{n}\,\underbrace{88\ldots 8}_{n-1}\,9, \quad n \geq 1\,.$$

Then

$$\begin{aligned} a_1 &= 49 = 7^2, \\ a_2 &= 4489 = 67^2 \end{aligned}$$

since

$$\begin{array}{rr|rr}
 & & 6 & 7 \\ \cline{3-4}
 & 6 & 44 & 89 \\
 & & 36 & \\ \cline{3-4}
12 & 7 & 8 & 89 \\
 & & 8 & 89 \\ \cline{3-4}
\end{array}$$

Also $a_3 = 444889 = 667^2$ since

$$\begin{array}{rr|rrr}
 & & 6 & 6 & 7 \\ \cline{3-5}
 & 6 & 44 & 48 & 89 \\
 & & 36 & & \\ \cline{3-5}
12 & 6 & 8 & 48 & \\
 & & 7 & 56 & \\ \cline{3-4}
132 & 7 & & 92 & 89 \\
 & & & 92 & 89 \\ \cline{4-5}
\end{array}$$

Can you work out the method used above to extract square roots of any number?

The above examples suggest we should consider

$$\begin{aligned}
b_s^2 &= (\underbrace{666\ldots 6}_{s}\,7)^2 \\
&= \left(\frac{200\ldots 01}{3}\right)^2 = \frac{1}{9}\Big(2\underbrace{00\ldots 0}_{s}1\Big)^2 \\
&= \frac{1}{9}\Big(4\underbrace{00\ldots 0}_{s}4\underbrace{00\ldots 0}_{s}1\Big) \\
&= \underbrace{44\ldots 4}_{s+1}\,\underbrace{88\ldots 8}_{s}9.
\end{aligned}$$

Hence the square root of a_{1000} is $\quad b_{999} = \underbrace{66\ldots 6}_{999}7.$

Q1099 I have three discs, each with one number on each side. Playing 3-up the totals 12, 13, 14, 15, 16, 17, 18 and 19 occur equally often. When the total was 18, the numbers on the three visible faces were 5, 6 and 7. Can you work out what numbers were written on the reverse sides? Is there just one solution, or are there several solutions? Is it possible, by starting with three suitably numbered discs, to obtain *any* eight consecutive numbers as totals in this way?

ANS.

$$\begin{array}{|c|}\hline 5\\\hline a\\\hline\end{array}\qquad\qquad\begin{array}{|c|}\hline 6\\\hline b\\\hline\end{array}\qquad\qquad\begin{array}{|c|}\hline 7\\\hline c\\\hline\end{array}$$

Suppose the first disc has 5 and a on opposite sides, the second disc has 6 and b, and the third disc has 7 and c on opposite sides. Since each of a, b, c, 5, 6, 7 occur $\frac{1}{2} \times 8 = 4$ times in the 8 totals, we have

$$12 + 13 + \cdots + 19 = 4 \times 31 = 4(5 + 6 + 7 + a + b + c).$$

So $a+b+c = 13$ or $c = 13-a-b$. Now $a+6+7 = 12, 14, 15, 16, 17$ or 19 so $a = -1, 1, 2, 3, 4$ or 6. Similarly $5+b+7 = 12, 14, 15, 16, 17$ or 19 so $b = 0, 2, 3, 4, 5$ or 7 and c is determined by the values of a and b, so we have reduced the problem to 36 cases to be checked. It is easy to check that

$$\begin{array}{|c|}\hline 5\\\hline 1\\\hline\end{array}\qquad\qquad\begin{array}{|c|}\hline 6\\\hline 4\\\hline\end{array}\qquad\qquad\begin{array}{|c|}\hline 7\\\hline 8\\\hline\end{array}$$

is one solution. Indeed, there are exactly six solutions: $(a, b, c) = (1, 4, 8)$, $(1, 7, 5)$, $(3, 2, 8)$, $(3, 7, 3)$, $(6, 2, 5)$ and $(6, 4, 3)$. I calculated these by hand and checked the solutions using a simple MAPLE program (see `www.maplesoft.com`).

To tackle the rider, consider adding 1 to each side of the 3 discs displayed above. We obtain

$$\begin{array}{|c|}\hline 6\\\hline 2\\\hline\end{array}\qquad\qquad\begin{array}{|c|}\hline 7\\\hline 5\\\hline\end{array}\qquad\qquad\begin{array}{|c|}\hline 8\\\hline 9\\\hline\end{array}$$

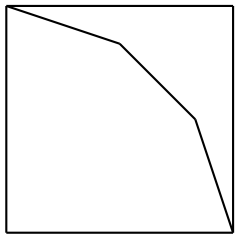
and the sums now run from 15 to 22. This method can be repeated to obtain six solutions running from $3n$ to $3n + 7$ for all n, positive and negative. If we consider sums running from 13 to 20 with 5, 6, 7 on one side of the discs we also find exactly six solutions $(a, b, c) = (1, 5, 9)$, $(1, 8, 6)$, $(4, 2, 9)$, $(4, 8, 3)$, $(7, 2, 6)$ and $(7, 5, 3)$. Similarly, for sums running from 14 to 21 we obtain $(a, b, c) = (1, 7, 9)$, $(1, 8, 8)$, $(6, 2, 9)$, $(6, 8, 3)$, $(7, 2, 8)$ or $(7, 7, 3)$.

Hence we have established that it is possible, by starting with three suitably numbered discs, to obtain any eight consecutive numbers as totals in this way.

Q1100 (a) $ABCD$ is a square. The point X is constructed so that

$$\angle BAX = \angle ABX = 15^\circ.$$

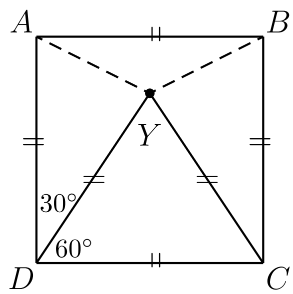
Show that $\triangle CDX$ is equilateral.
(b)



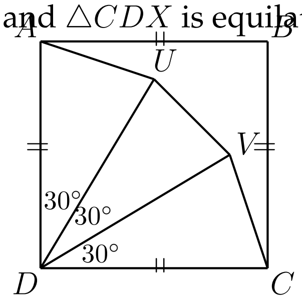
A square is constructed on a regular dodecagon (twelve sided polygon) so that one vertex is at the centre of the dodecagon and two vertices of the square coincide with two vertices of the dodecagon, as in the diagram.

Using part (a) of this problem, calculate the ratio of the area of the square to the area of the part of the dodecagon it covers.

ANS.
(a)

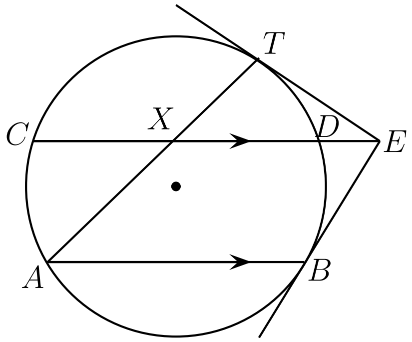


Let Y be the point inside the square $ABCD$ such that $\triangle CDY$ is equilateral. Then $AB = BC = CD = DA = DY = CY$. Also $\angle ADY = 90° - \angle CDY = 30°$. But $\triangle ADY$ is isosceles so $\angle DAY = \frac{1}{2}(180° - 30°) = 75°$.
Thus $\angle BAY = 15° = \angle ABY$, by symmetry of the argument. However, X is defined by $\angle BAX = \angle ABX = 15°$ so $X = Y$ and $\triangle CDX$ is equilateral.
(b)



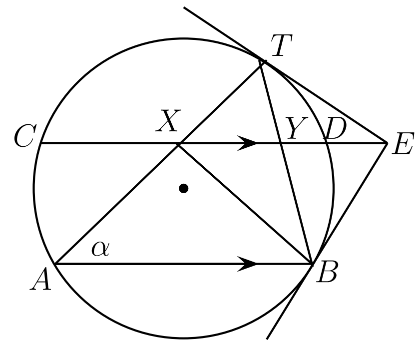
In part (a) the area of $\triangle AXD = \frac{1}{2} \times 1 \times \sin 30° = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ if the square has edge length 1. The triangles $\triangle AUD$, $\triangle UVD$ and $\triangle VCD$ are each congruent to $\triangle AYD$ in (a). Hence the area of $AUVCD$ is $\frac{3}{4}$ and the ratio required is $\frac{4}{3}$. (Why can we assume the square has side length 1.)

Q1101



Two chords AB and CD in a circle are parallel. The tangent at B meets CD produced at E. The other tangent which passes through E meets the circle at T. AT meets CD at X. Show that $CX = XD$.

ANS. Join TB to meet CE at Y and also join XB. Let $\angle TAB = \alpha$.

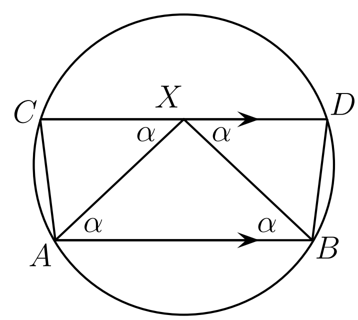


$\angle TXE = \alpha$ since $CD \parallel AB$ and $\angle ETB = \angle TAB = \alpha$ by the alternate segment theorem. Next $ET = EB$ since they are both tangents from E to the circle, so $\triangle ETB$ is isosceles and $\angle EBT = \alpha$.

$\triangle XTE$ is similar to $\triangle TYE$ since $\angle TXE = \angle YTE = \angle BTE = \alpha$ and $\angle E$ is common.

Hence $\frac{TE}{XE} = \frac{YE}{TE}$ or $TE^2 = XE \times YE$ and $BE^2 = TE^2 = XE \times YE$.

We now consider $\triangle EBX$ and $\triangle EYB$. These are similar since $\angle E$ is common and $\frac{EB}{EY} = \frac{EX}{EB}$ shows corresponding sides have equal ratio. Hence $\alpha = \angle EBY = \angle EXB$ and $\angle ABX = \alpha$, alternate angles. By this stage we have six angles equal to α



and in particular $\triangle XAB$ is isosceles, so $XA = XB$.

To complete the proof join AC and BD and let $\angle XCA = \delta$. Then $\angle ABD = 180° - \delta$ (cyclic quadrilateral), $\angle XBD = 180° - \delta - \alpha$, so $\angle XDB = \delta$. Finally $\triangle CXA \equiv \triangle DXB$ and $CX = XD$.

Q1102 Find the first positive integer whose square ends in three 4s. Find all positive integers whose square ends in three 4s. Show that no perfect square ends in four 4s.

ANS. A number squared ends in a 4 if and only if it ends in a 2 or an 8. That is, $x^2 \equiv 4 \pmod{10}$ iff $x \equiv 2$ or $8 \pmod{10}$. We proceed by considering the last 2 digits, the last 3 digits, etc.

Suppose $x = 10a + 2$. Then $x^2 = 100a^2 + 40a + 4 \equiv 40a + 4 \pmod{100}$, so $a \equiv 1$ or $6 \pmod{10}$ if $x^2 \equiv 44 \pmod{100}$. Similarly if $x = 10a - 2$ then

$$x^2 = 100a^2 - 40a + 4 = 100(a^2 - a) + 60a + 4 \equiv 60a + 4 (\mathrm{mod} 100)$$

and $a \equiv 4$ or $9 \bmod 10$.
Overall $x \equiv 12, 38, 62$ or $88 \equiv 12, 50 - 12, 50 + 12$ or $100 - 12 \pmod{100}$.

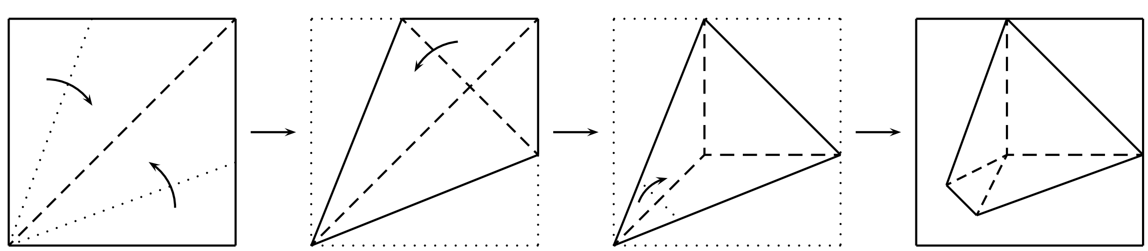
Next suppose $x = 50b + 12$ or $50c - 12$ and $x^2 \equiv 444 \pmod{1000}$. Then $x^2 = 2500b^2 + 1200b + 144$ so b is odd. Similarly c is odd so $x = 100d + 38$ or $100e + 62$. Since $(100d + 38)^2 \equiv 38 \times 200d + 38^2 \pmod{1000} \equiv 600d + 444 \pmod{1000}$, d is a multiple of 5 and $x = 500f + 38$. We can similarly show that e is a multiple of 5. Finally

$$(500f \pm 38)^2 = 25000f^2 \pm 38000f + 1444 \equiv 1444 \pm 8000f \pmod{10^4} \not\equiv 4444 \pmod{10^4}$$

for any value of f.

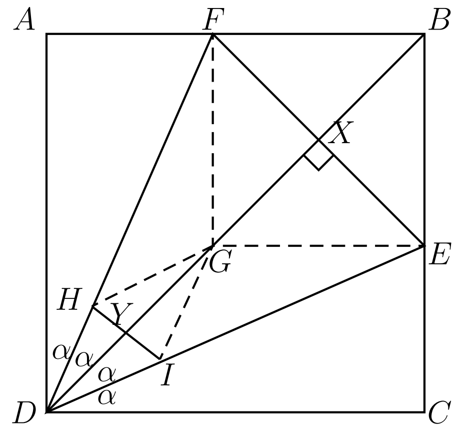
In summary if $x = 500\alpha \pm 38$ then x^2 ends in three 4's for all values of α but does not end in four 4's for any value of α.

Q1103 In households all over Australia squares of cloth are often folded as follows:



What is the ratio of the area of the folded area to the area of the original square?

ANS.



Without loss of generality we consider a unit square $ABCD$ with DE and DF angle bisectors of $\angle CDB, \angle ADB$ respectively. X is the point where BD meets EF, $BEGF$ is a square and IYH is the perpendicular bisector of DG.

Let $\alpha = 22\frac{1}{2}^\circ$ and to facilitate the calculations let $t = \tan\alpha$.

Clearly $\angle CDE = \angle XDE = \angle XDF = \angle ADF = \alpha.$

Also $\triangle CDE \equiv \triangle XDE \equiv \triangle XDF \equiv \triangle ADF.$

Next some easy length calculations: if $CE = t$, then $BE = 1 - t$,

$$\begin{aligned} FE &= BG = (1-t)\sqrt{2}, \\ GD &= \sqrt{2} - (1-t)\sqrt{2} = t\sqrt{2}, \\ DY &= t\sqrt{2}/2, \\ DX &= DG + BG/2 = t\sqrt{2} + (1-t)\sqrt{2}/2 = (1+t)/\sqrt{2}. \end{aligned}$$

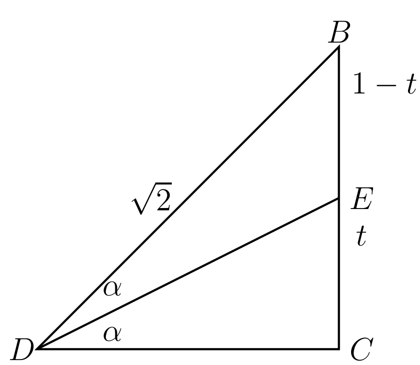
The area of each of the above 4 congruent triangles is $t/2$ hence area of $\triangle DEF = t$. Next, by similar triangles,

$$\frac{\text{Area}\,\triangle DHI}{DY^2} = \frac{\text{Area}\triangle DFE}{DX^2} = \frac{t}{(1+t)^2/2}$$

$$\text{Area}\,\triangle DHI = \frac{t^3}{(1+t)^2}$$

Next

$$\text{Area}EFHI = \text{Area}DFE - \text{Area}DHI = t - \frac{t^3}{(1+t)^2} = \frac{2t^2+t}{(1+t)^2}$$



To calculate t we use the angle bisector theorem

$$\frac{1}{t} = \frac{\sqrt{2}}{1-t}, \quad t\sqrt{2} = 1 - t, \quad t = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1$$

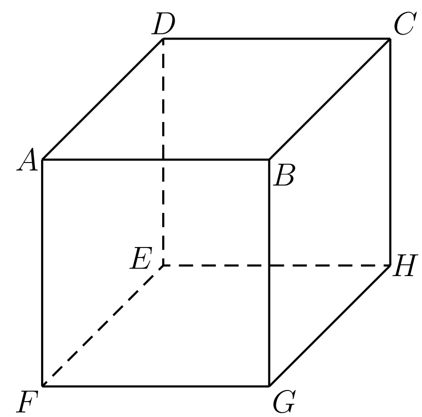
and

$$\text{Area}EFHI = \frac{2(\sqrt{2}-1)^2 + \sqrt{2} - 1}{2} = \frac{5 - 3\sqrt{2}}{2} \cong .379 \, .$$

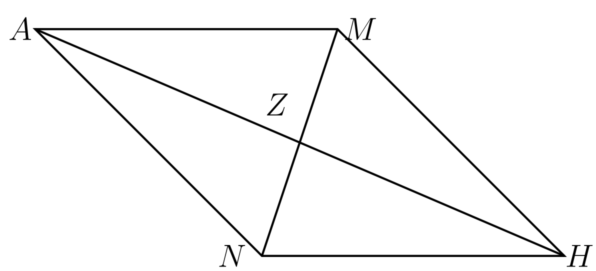
Q1104 $ABCDEFGH$ is a cube of side 2.

(a) Find the area of the quadrilateral $AMHN$, where M is the midpoint of BC, and N is the midpoint of EF.

(b) Let P be the midpoint of AB, and Q the midpoint of HE. Let AM meet CP at X, and let HN meet FQ at Y. Find the length of XY.

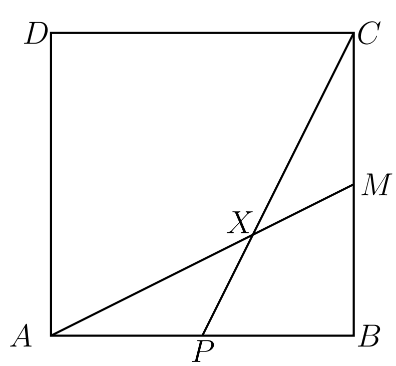


ANS.



(a) $AM^2 = MH^2 = HN^2 = NA^2 = 5$ and $AH^2 = 2^2 + 2^2 + 2^2 = 12$. So $AM = \sqrt{5}$, $AZ = \sqrt{3}$ and $ZM = \sqrt{2}$. So area $AMHN = 2 \times \frac{1}{2} \times 2\sqrt{3}\sqrt{2} = 2\sqrt{6}$.

(b)



One way to tackle this problem is to use coordinates in 3 dimensions. We fix $F = (0,0,0)$ as the origin, $G = (2,0,0), E = (0,2,0)$ and $A = (0,0,2)$. Hence $B = (2,0,2), D = (0,2,2), C = (2,2,2), P = (1,0,2)$ and $M = (2,1,2)$.

AM and CP are both medians of $\triangle ABC$ so they trisect each other. Hence $X = (4/3, 2/3, 2)$. An analogous calculation in the $z = 0$ plane gives $Y = (2/3, 4/3, 0)$. Hence

$$XY^2 = \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 2^2 = \frac{44}{9} \quad \text{and} \quad XY = \frac{2\sqrt{11}}{3}.$$