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THE 'HAPPY END' PROBLEM

David Harvey[1]

The 'Happy End' problem is a geometrical problem that has permeated George Szekeres' life. It was first posed by Esther Klein in 1933. Their colleague Paul Erdös named it the 'Happy End' problem, because it led to George and Esther's marriage in 1937. It also inspired a joint publication, 'A Combinatorial Problem in Geometry' (1935), written by George Szekeres and Paul Erdös. George is working on this problem even today – after all, it hasn't been completely solved yet!

To describe the problem, we need a little terminology. If we have a collection of points in the plane, the points are said to be in *general position* if no three of them are collinear. Figure (1a) shows a collection of points which are in general position; Figure (1b) shows a collection which is not in general position.

Figure (1a)

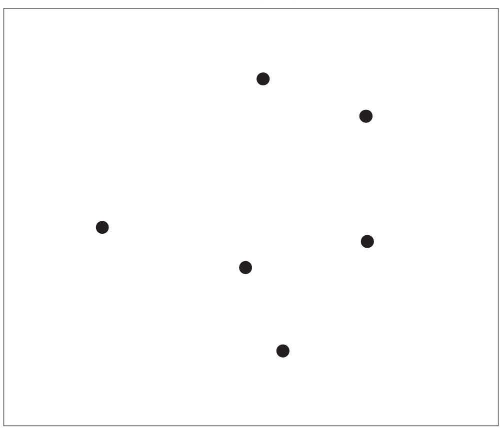
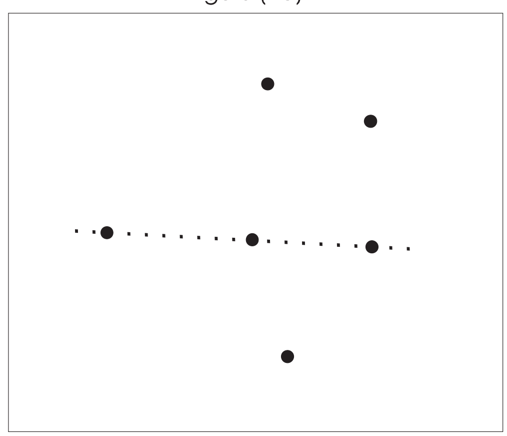


Figure (1b)



Also, a polygon is said to be *convex*, if any two points inside the polygon can be joined by a line segment which does not fall outside the polygon. Figure (2a) shows a convex polygon. The polygon in Figure (2b) is not convex, because the line segment joining the points A and B does not lie inside the polygon. A looser but more intuitive definition of 'convex' is that the polygon has no 'indentations'.

[1]This is a follow-on to David's biography of George Szekeres in the previous article.

Figure (2a)

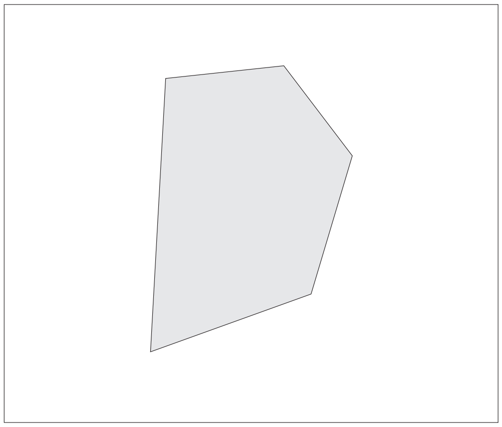
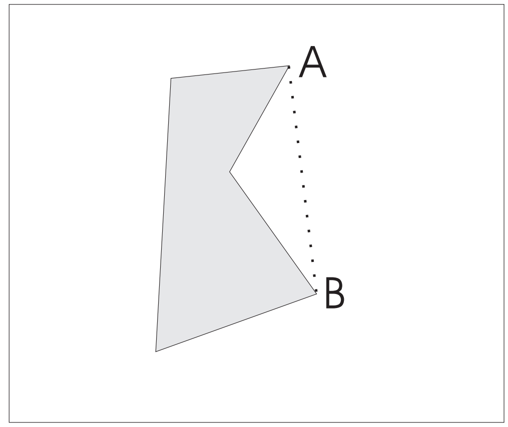


Figure (2b)



Now we can state Esther's original problem. Suppose you have five points in general position in the plane. Then, it is always possible to find four of these points which form a convex quadrilateral.

Esther's elegant proof of this fact proceeds as follows. First we have to construct the 'convex hull' of the five given points. The easiest way to visualise this is to imagine that the points represent nails sticking out of a wooden board, and that we stretch an elastic band around the nails. The convex hull is the polygon formed by the elastic band.

In Figure (3a), the boundary of the convex hull includes all five points, so it is a pentagon. In this case, we can choose any four points we like, and they will form a convex quadrilateral.

Figure (3a)

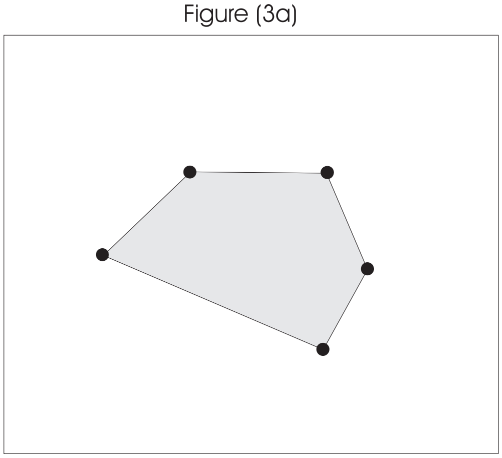
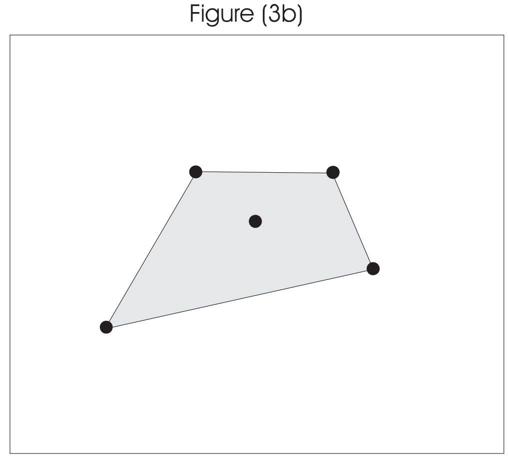
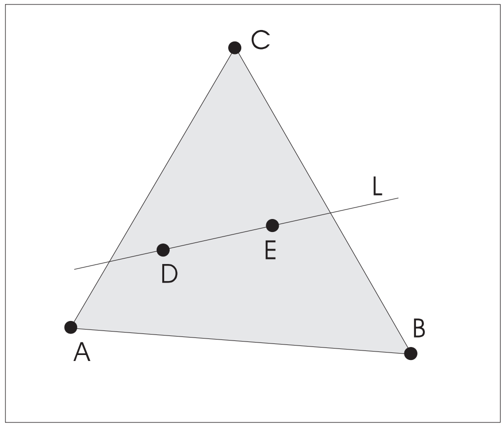


Figure (3b)



In Figure (3b), the boundary of the convex hull only includes four of the points, with the last remaining point strictly inside the convex hull. In this case, we obviously choose the four points on the outside to be our convex quadrilateral.

Figure (3c)



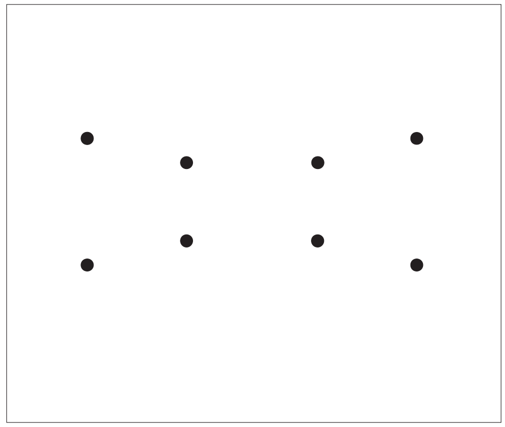
In Figure (3c), we have the trickiest situation. Here the convex hull only includes three of the points (say A, B, and C), with the remaining two points (say D and E) strictly inside the convex hull. Let L be the line joining D and E. Since the five points are in general position, the line L cannot include any of A, B or C. It is fairly easy to see that two of the external points (in this case, A and B) must lie on one side of L, and the other point (in this case C) must lie on the other. Therefore, the two interior points, and the two points which lie on the same side of L, together form a convex quadrilateral, as shown in Figure (3c). This completes the proof!

In their 1935 paper, George and Paul proved a generalisation of Esther's result. They showed that if you choose any number $n > 2$, then there exists a number m, such that for any collection of m points in general position in the plane, you can always find n of those points which form a convex n-gon. (An *n-gon* is a polygon with n sides. For example, a 4-gon is a quadrilateral.)

Esther's problem is a special case of this, when $n = 4$. In this case, the best possible value of m is 5, since for every five points in general position, you can always find four of them which form a convex 4-gon.

When $n = 5$, it is known that the best possible value of m is 9. This means that from any collection of nine points in general position in the plane, you can always choose five which form a convex 5-gon. Figure (4) shows that eight points are not enough to ensure that there is a convex 5-gon.

Figure (4)



For $n > 5$, nobody knows what the best possible value of m is. In their 1935 paper, George and Paul proved that the best value of m lies in the range:

$$2^{n-2} + 1 \leq m \leq \frac{(2n-4)!}{(n-2)!^2}.$$

No progress was made for about the next sixty years. Then, during 1998, several successive breakthroughs were made, and the upper bound was reduced substantially. The most recent result is due to Géza Toth and Pavel Valtr, closely following work by Daniel Kleitman, Lior Pachter, Fan Chung, and Ron Graham.

Most researchers in this area believe that $2^{n-2} + 1$ is the true value. Some have even offered a (token) cash reward for a proof of this fact!

These days, if you drop in on George in his office at UNSW, you will most likely find him tapping away at his computer, still working on the 'Happy End' problem. He is currently writing a program which will carry out an exhaustive search of the case $n = 6$. Everyone suspects that the correct value of m is 17 in this case. The program is designed to examine every possible arrangement of 17 points, and check that each one does indeed contain a convex 6-gon. It will run on hundreds of computers in parallel, and George believes even with this kind of computing capacity, the calculation will take several weeks.