Parabola Volume 38, Issue 2 (2002)

UNSW SCHOOL MATHEMATICS COMPETITION 2002

SOLUTIONS

JUNIOR DIVISION

1. We have four red cards, four blue, four yellow and four green. Is it possible to arrange 15 of these in a 3×5 rectangle in such a way that no two adjacent cards (horizontally, vertically or diagonally) have the same colour?

   First solution:
   No! If we try to lay out the cards as suggested, there are two possibilities.

   If the three cards in the first column have different colours, A, B and C in the order ABC then the second column is CDA, the third is ABC, the fourth is CDA and the fifth would have to be ABC. This requires five A's and five C's, which we don't have.

   If the first column uses only two different colours, A and B in the order ABA then the second is CDC, the third is ABA or BAB, the fourth is CDC or DCD, and the fifth would have to be ABA or BAB. In any case, the first, third and fifth columns require nine cards of the colours A and B, which we don't have.

   Second solution: No! Number the positions in the first row 1 to 5, in the second 6 to 10, and in the third 11 to 15 (each from left to right).

   Suppose the card in position 7 is Red. Then none of 1,2,3,6,8,11,12 and 13 is Red. If either 9 or 10 is Red then no other position is Red, and there are thirteen positions to be filled by the twelve cards that are not Red.

   So one of positions 4 and 5, and one of positions 14 and 15, is Red, and so precisely three positions are Red. Similarly, if position 9 is Green, precisely three positions are Green. So at most six positions are Red or Green, and there are nine positions to be filled by the eight cards that are not Red or Green.

2. Prove that the sequence

   $$1,\ 61,\ 661,\ 6661,\ 66661,\ \cdots$$

   contains infinitely many numbers divisible by 7.

Solution:
First, it is easy to check that $66661 = 7 \times 9523$. Also $666666 = 666 \times 1001 = 666 \times 7 \times 11 \times 13$. So, $666 \cdots 61$ is divisible by 7 whenever the number of 6's is $4 + 6m$, where m is a non–negative integer.

3. We start with the set of numbers $1,\ 1/2,\ 1/3,\ \cdots\ 1/2002$, and we perform the following operation: We are allowed to delete any two of them, say a and b, and replace them with the single number $ab + a + b$. After performing 2001 such operations, only one number remains. Prove that this number is the same, no matter in which order we perform the operations, and find the number.

   Solution:
   Suppose we start with a set of n numbers, $a_1, a_2, \cdots, a_n$. It is easy to prove by induction on n that for $n \geq 2$, we end up with the Number $(a_1+1)(a_2+1)\cdots(a_n+1) - 1$. Thus if $n = 2002$ and $a_k = \frac{1}{k}$, we end up with

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{2002}\right) - 1 = 2002.$$

4. One million clever high school students from all over the world are invited to participate in the 2002 UNSW School Mathematics Competition. Each of them is assigned a unique 6-digit student number ranging from 000000 to 999999. A student number is thought to be lucky if none of its digits is 9 and the sum of the first three digits is equal to the sum of its last three digits. For instance, 003111 and 468567 are both lucky. (i) Show that the sum of all lucky student numbers is divisible by 2002. (ii) How many lucky student numbers are there?

   Solution:
   If a is a digit in $\{0, \cdots, 8\}$, let $\overline{a} = 8 - a$. If $abcdef$ is a lucky student number, then so is $\overline{a}\overline{b}\overline{c}\overline{d}\overline{e}\overline{f}$, and they add to 888888. Moreover, they are different unless they are both 444444.

   (i) The sum of the lucky numbers is thus a multiple of $444444 = 222 \times 2002$.

   (ii) The number of lucky student numbers with the sum of the first three digits equal to 4 is $15^2 = 225$, since the first three digits could be any of the following 15 possibilities:

$$400,\ 040,\ 004,\ 310,\ 301,\ 130,\ 103,\ 031,\ 013,\ 220,\ 202,\ 022,\ 211,\ 121,\ 112,$$

   and so could the last three digits.

   In the same way (by listing all possibilities), we find that the number of lucky student numbers with the sum of the first three digits equal to k ($k = 0, 1, \cdots, 24$) is given by the following table:

k	0	1	2	3	4	5	6	7	8	9	10	11	12
	1^2	3^2	6^2	10^2	15^2	21^2	28^2	36^2	45^2	52^2	57^2	60^2	61^2

for k up to and including 12, while for $k > 12$, the number of lucky student numbers is the same for k as it is for $24-k$, again by virtue of the fact that if $abcdef$ is a lucky student number so is $\overline{a}\overline{b}\overline{c}\overline{d}\overline{e}\overline{f}$. So the number of lucky student numbers is

$$\begin{aligned}
&2\left(1^2+3^2+6^2+10^2+15^2+21^2+28^2+36^2+45^2+52^2+57^2+60^2\right)+61^2\\
&\quad=32661
\end{aligned}$$

and the sum of all the lucky student numbers is

$$16330\times 888888+444444=14515985484=7250742\times 2002.$$

5. In the acute–angled triangle ABC, the perpendicular from A onto BC meets BC at D, the perpendicular from C onto AB meets AB at E. The length of AE is 5, the length of BE is 3, the length of BD is 2 and the length of CD is x. Find x.

   First solution:
   This uses Pythagoras's equation and areas.

$$\begin{aligned}
AD&=\sqrt{AB^2-BD^2}=\sqrt{8^2-2^2}=\sqrt{60}=2\sqrt{15},\\
CE&=\sqrt{CB^2-BE^2}=\sqrt{(x+2)^2-3^2}.
\end{aligned}$$

$$\begin{aligned}
\text{Area of }\Delta ABC&=\frac{1}{2}\times AD\times BC=\sqrt{15}(x+2),\\
&=\frac{1}{2}\times CE\times AB=\frac{1}{2}\times\sqrt{(x+2)^2-3^2}\times 8=4\sqrt{(x+2)^2-9}.
\end{aligned}$$

   Thus we find

$$\begin{aligned}
\sqrt{15}(x+2)&=4\sqrt{(x+2)^2-9},\\
15(x+2)^2&=16\left((x+2)^2-9\right),\\
(x+2)^2&=16\times 9,\\
x+2&=4\times 3=12,\\
x&=10.
\end{aligned}$$

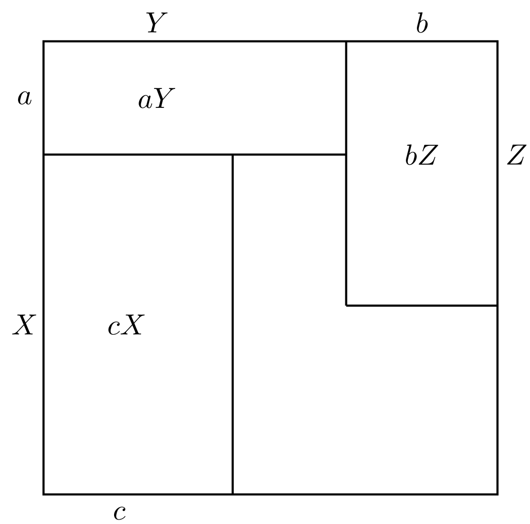
   Second solution: This uses similar triangles.

   Δ's $ABD,\ CBE$ are similar since $\angle ADB=\angle CEB$ (rightangles) and $\angle ABD=\angle CBE$ (common). So

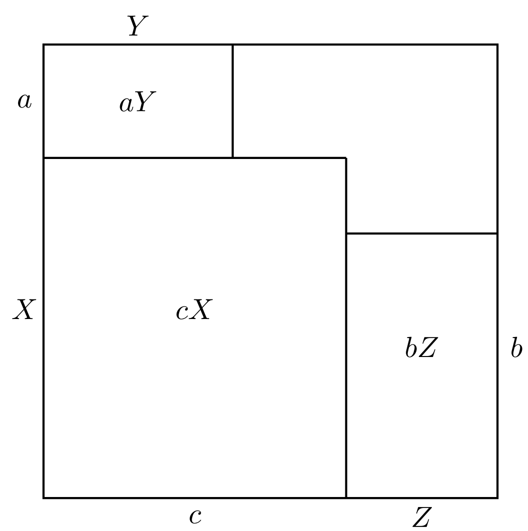
$$\begin{aligned}
\frac{x+2}{3}&=\frac{8}{2},\\
x+2&=12,\\
x&=10.
\end{aligned}$$

6. The numbers $a,\ b,\ c,\ X,\ Y,\ Z,\ k$ are all positive, and $a+X=b+Y=c+Z=k$. Show that $aY+bZ+cX<k^2$.

First solution: (geometric)
Suppose $c \le Y$. We have



Now suppose $c > Y$. We have



In either case, $aY + bZ + cX < k^2$.

Second solution: (algebraic)

$$\begin{aligned} k^3 &= (a+X)(b+Y)(c+Z) \\ &= abc + XYZ + aY(c+Z) + bZ(a+X) + cX(b+Y) \\ &= abc + XYZ + k(aY + bZ + cX) \\ &> k(aY + bZ + cX). \end{aligned}$$

The result follows.

Third solution: (algebraic)

$XYZ > 0$ so $(k-a)(k-b)(k-c) > 0$, or,

$$\begin{aligned}
&k^3 - k^2(a+b+c) + k(ab+bc+ca) - abc > 0,\\
&k^3 > k^2(a+b+c) - k(ab+bc+ca),\\
&k^2 > k(a+b+c) - (ab+bc+ca),\\
&k^2 > a(k-b) + b(k-c) + c(k-a),\\
&k^2 > aY + bZ + cX.
\end{aligned}$$

SENIOR DIVISION

1. We start with the set of numbers $1,\ 1/2,\ 1/3,\ \cdots\ 1/2002$, and we perform the following operation: We are allowed to delete any two of them, say a and b, and replace them with the single number $ab + a + b$. After performing 2001 such operations, only one number remains.

   Prove that this number is the same, no matter in which order we perform the operations, and find the number.

   Solution:
   See Junior 3.

2. Determine the value of

$$\binom{2002}{0} - \binom{2001}{1} + \binom{2000}{2} - \cdots - \binom{1001}{1001}.$$

   Solution:
   Let

$$F_n = \binom{n}{0} - \binom{n-1}{1} + \binom{n-2}{2} - + \cdots .$$

   Then $F_0 = 1,\ F_1 = 1$ and since

$$\binom{n-k}{k} = \binom{n-k-1}{k-1} + \binom{n-k-1}{k},$$

   it follows that

$$F_n = F_{n-1} - F_{n-2}.$$

   Thus the sequence $\{F_n\},\ n = 0, 1, \cdots$ is $\{1, 1, 0, -1, -1, 0, 1, 1,\ \cdots\}$. That is, the sequence is periodic, with period 6. Since $2002 = 6 \times 333 + 4$, $F_{2002} = F_4 = -1$.

3. The numbers $a,\ b,\ c,\ X,\ Y,\ Z,\ k$ are all positive, and $a + X = b + Y = c + Z = k$. Show that $aY + bZ + cX < k^2$.

   Solution:
   See Junior 6.

4. $UNSW$ is a square. The points A and B lie on UN and UW respectively and $UA = UB$. Also, E lies on BN and UE is perpendicular to BN. Prove that $\angle SEA = 90^{\circ}$.

   First solution: (using trigonometry and the converse of Pythagoras's theorem).
   Let $\angle UNB = \theta$, and the side of the square be s. Drop perpendiculars from E onto

UN and UW, to meet UW at X and UN at Y. Then $UB = s\tan\theta$, $UA = s\tan\theta$, $UE = s\sin\theta$, $\angle BUE = \theta$, $EX = s\sin\theta\sin\theta = s\sin^2\theta$, $EY = s\sin\theta\cos\theta$ and

$$\begin{aligned}
&SA^2 - (SE^2 + AE^2)\\
&= s^2 + (s - s\tan\theta)^2\\
&- \big((s - s\sin\theta\cos\theta)^2 + (s - s\sin^2\theta)^2 + (s\tan\theta - s\sin^2\theta)^2 + (s\sin\theta\cos\theta)^2\big)\\
&= 2s^2\left(\sin\theta\cos\theta + \sin^2\theta + \tan\theta\sin^2\theta - \sin^2\theta\cos^2\theta - \sin^4\theta - \tan\theta\right)\\
&= 2s^2\left(\sin\theta\cos\theta + \tan\theta\sin^2\theta - \tan\theta(\sin^2\theta + \cos^2\theta)\right)\\
&= 2s^2\left(\sin\theta\cos\theta - \tan\theta\cos^2\theta\right)\\
&= 0.
\end{aligned}$$

It follows that $\angle SEA$ is a rightangle.

Second solution: (using coordinate geometry) Let U be $(0,0)$, N be $(0,s)$, S be (s,s), W be $(s,0)$, A be $(0,a)$, B be $(a,0)$.

The equation of NB is $\dfrac{x}{a} + \dfrac{y}{s} = 1$, or, $sx+ay = as$. The equation of UE is therefore $ax - sy = 0$.

E is $\left(\dfrac{as^2}{a^2+s^2}, \dfrac{a^2s}{a^2+s^2}\right)$.

The slope of SE is

$$\frac{s - \frac{a^2s}{a^2+s^2}}{s - \frac{as^2}{a^2+s^2}} = \frac{s^3}{a^2s + s^3 - as^2} = \frac{s^2}{a^2+s^2-as}.$$

The slope of AE is

$$\frac{\frac{a^2s}{a^2+s^2} - a}{\frac{as^2}{a^2+s^2}} = \frac{a^2s - a^3 - as^2}{as^2} = \frac{as - a^2 - s^2}{s^2}.$$

The product of the slopes is -1, so $\angle SEA$ is a rightangle.

Third solution: (using similar triangles). It is easy to see that Δ's UBE and NUE are similar.

It follows that

$$\frac{UB}{UE} = \frac{NU}{NE}.$$

Now, $UB = UA$, $NU = NS$, so

$$\frac{UA}{UE} = \frac{NS}{NE}.$$

It now follows that Δ's UEA and NES are similar, so $\angle UEA = \angle NES$.

If we add $\angle NEA$, we find that $\angle SEA = \angle NEU$, which is a rightangle.

Fourth solution: (using circle geometry) Produce (that is, extend) UE to intersect SW at C. Δ's UWC and NUB are congruent, so $WC = UB = UA$. So $ANSC$ is a rectangle. Let O be the point of intersection of the diagonals of $ANSC$. Then $OA = ON = OS = OC$. Construct a circle centred at O with radius OA. This circle passes through E since $\angle NEC$ is a rightangle (and NC is a diameter). But then $\angle SEA$ is an angle at the circumference (SA is a diameter), so is a rightangle.

5. Let OX, OY denote the positive x and y axes. The rigid triangle ABC has a rightangle at C. It glides in the plane, A on OY, B on OX, while C is on the opposite side of AB to O. (i) What is the path described by the midpoint of AB? (ii) What is the path described by C?

   Solution:
   (i) The distance of the midpoint of AB from the origin is constant and equal to $\frac{1}{2}AB$. It follows that the locus of the midpoint of AB is an arc of a circle centred at the origin, radius $\frac{1}{2}AB$.

   (ii) Since $\angle ACB$ and $\angle AOB$ are both rightangles, $ACBO$ is a cyclic quadrilateral. It follows that $\angle COB$ is equal to $\angle CAB$, which is constant, and equal to α, say. It follows that the locus of C is part of the straight line which makes an angle α with the positive x-axis.

6. The numbers $x_1, x_2, x_3, \cdots, x_{2001}, x_{2002}$ are positive. Prove that

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots + \frac{x_{2001}^2}{x_{2002}} + \frac{x_{2002}^2}{x_1} \geq x_1 + x_2 + \cdots + x_{2001} + x_{2002}.$$

   First Solution:
   We can prove by induction on n that for $n \geq 2$, if $x_1, x_2, \cdots, x_n$ are positive, then

$$\frac{x_1^2}{x_2} + \cdots + \frac{x_n^2}{x_1} \geq x_1 + \cdots + x_n.$$

   (Call this proposition P_n).

   We must first check that P_2 is true. That is, that

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_1} \geq x_1 + x_2.$$

   This is equivalent to

$$(x_1 + x_2)(x_1 - x_2)^2 \geq 0,$$

   which is true. Now suppose P_n true, and that $x_1, x_2, \cdots, x_n, x_{n+1}$ are positive. We can suppose that x_{n+1} is the smallest, since, if not, we can permute the numbers cyclically so that x_{n+1} is the smallest. (This permutation does not alter either side of the purported inequality.) Since P_n is true, we know that

$$\frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} + \frac{x_n^2}{x_1} \geq x_1 + \cdots + x_{n-1} + x_n.$$

We subtract from this the term

$$\frac{x_n^2}{x_1}$$

and add

$$\frac{x_n^2}{x_{n+1}} + \frac{x_{n+1}^2}{x_1}$$

to obtain

$$\frac{x_1^2}{x_2} + \cdots + \frac{x_{n+1}^2}{x_1} \geq x_1 + \cdots + x_n - \frac{x_n^2}{x_1} + \frac{x_n^2}{x_{n+1}} + \frac{x_{n+1}^2}{x_1}.$$

We now need to show that the right hand side is greater than or equal to $x_1 + \cdots + x_{n+1}$. This is equivalent to showing that

$$-\frac{x_n^2}{x_1} + \frac{x_n^2}{x_{n+1}} + \frac{x_{n+1}^2}{x_1} \geq x_{n+1}$$

or,

$$(x_n^2 - x_{n+1}^2)(x_1 - x_{n+1}) \geq 0,$$

which is true.

P_2 is true, and if P_n is true then P_{n+1} is true. So P_n is true for $n \geq 2$ by induction.

Second solution: Since $(a-b)^2 \geq 0$, $a^2 + b^2 \geq 2ab$. If b is positive,

$$\frac{a^2}{b} + b \geq 2a.$$

Therefore

$$\frac{x_1^2}{x_2} + x_2 \geq 2x_1, \quad \frac{x_2^2}{x_3} + x_3 \geq 2x_2, \quad \cdots \quad \frac{x_n^2}{x_1} + x_1 \geq 2x_n.$$

If we add all these inequalities and subtract $x_1 + x_2 + \ \cdots \ + x_n$, we obtain

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \ \cdots \ + \frac{x_n^2}{x_1} \geq x_1 + x_2 + \ \cdots \ + x_n.$$

WEB HIGHLIGHTS

http://www-history.mcs.st-andrews.ac.uk/history/index.html

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