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PROBLEM SECTION

You are invited to submit solutions to any or all of the following problems, accompanied by your name, school and year. Solutions to these problems will appear in the next issue of *Parabola*; your solution(s) may be used if they are received in time.

Q1121 Prove that $g(x) = 30x^3 + 80x^2 + 72x + 66$ is irreducible over the integers by showing it has no roots modulo 13. What are the roots of $g(x)$ modulo 7?

Q1122 Given two full 10-litre flasks of mercury, an empty 5-litre flask and an empty 4-litre flask but no other container capable of holding mercury, how can you put 3 litres of mercury in each of the smaller flasks leaving a total of 14 litres in the larger flasks?

Q1123 Check that $\cos 2\theta = 2\cos^2\theta - 1$ and that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. Now show that 1, $\cos 72°$ and $\cos 144°$ are the roots of the cubic equation $4x^3 - 3x = 2x^2 - 1$, and so evaluate $\cos 72°$ as a surd.

Q1124 Use your answer to the above question to show how an angle of $72°$ may be constructed in the plane using ruler and compasses only.

Q1125 Use the second formula in Q1122 to obtain a formula for the solutions of the cubic equation $4x^3 - 3k^2x = a$ valid for a range of values of the real constants k and a. What is the range of values for which there are three real roots? Use a pocket calculator to find the three roots of $4x^3 - 2.9x = 0.5$.

Q1126 Into how many pieces can a disc be cut with 2 straight lines? Ditto 3 straight lines, 4 straight lines, 5 straight lines, 6 straight lines? How can your answer be generalised?

Q1127 Arrange 11 points in the plane so that there are 16 lines each of which contains 3 of these points.

Q1128 [HARD] Here a cyclic quadrilateral Q is a convex quadrilateral whose vertices lie on a circle. There is an ancient formula for the area $|Q|$ of a cyclic quadrilateral with sides of lengths w, x, y, z. Set $s = \frac{1}{2}(w+x+y+z)$. Then $|Q|^2 = (s-w)(s-x)(s-y)(s-z)$. This problem asks you to verify directly the trigonometric identity underlying the formula and thus to prove it. Take the circle containing the vertices of Q to have unit radius and let the four isoceles triangles whose vertices are the centre of the circle and two adjacent vertices of the quadrilateral have angles $2A$, $2B$, $2C$ and $2D$ at their common vertex. Thus $A + B + C + D = 180°$. So here the side lengths may be taken to be $2 \sin A$, $2 \sin B$, $2 \sin C$ and $2 \sin D$ and the area of Q is $\frac{1}{2}(\sin 2A + \sin 2B + \sin 2C + \sin 2D)$. You are being asked to prove — or check electronically — that if $A + B + C + D = 180°$ then

$$\begin{aligned} & \frac{1}{4}(\sin 2A + \sin 2B + \sin 2C + \sin 2D)^2 \\ = \; & (-\sin A + \sin B + \sin C + \sin D) \cdot (\sin A - \sin B + \sin C + \sin D) \cdot \\ & (\sin A + \sin B - \sin C + \sin D) \cdot (\sin A + \sin B + \sin C - \sin D)\,. \end{aligned}$$

Q1129 Show how to deduce the Heron formula for the area of a triangle with sides of lengths a, b, c from the above formula for the area of a cyclic quadrilateral.

Q1130 Show that there is a cyclic quadrilateral with sides having different integer lengths in metres such that its area is an integer number of square metres.