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SOLUTIONS TO PROBLEMS 1111-1120

Q1111 A grandfather clock takes 30 seconds to strike 6 o'clock. How long does it take to strike 12 o'clock?

ANS. This is an old trick puzzle. The striking is taken to be instantaneous, and so there must be 6 seconds between strikes. Hence 66 seconds are needed for the clock to strike 12.

Q1112 Simplify the following expression:

$$\frac{(2^3-1)(3^3-1)\dots(2002^3-1)}{(2^3+1)(3^3+1)\dots(2002^3+1)}.$$

ANS. A non-intellectual approach to this sort of problem is to run it in a symbolic calculating package. The following is output from Maple:

```
A:=1:
for n from 2 to 2002 do
A:=simplify((A*(n^3-1))/(n^3+1)):
od:
A;
                    1336669
                    -------
                    2005003
quit
```

Vinoth Nandakumar of Sydney Boys High, who also did useful work on certain other problems, used the standard factorisations of x^3-1 and x^3+1 to write the required product P as

$$P = \frac{(2-1)(2^2+2+1)(3-1)(3^2+3+1)}{(2+1)(2^2-2+1)(3+1)(3^2-3+1)} \times \cdots \frac{(2002-1)(2002^2+2002+1)}{(2002+1)(2002^2-2002+1)}$$

He then used the fact that $(x+1)^2-(x+1)+1 = x^2+x+1$ to cancel out numerous terms above to get:

$$\begin{aligned} P &= \frac{(2-1)(3-1)(4-1)\dots(2002-1)(2002^2+2002+1)}{(2+1)(2^2-2+1)(3+1)(4+1)\dots(2002+1)} \\ &= \frac{(2-1)(3-1)(2002^2+2002+1)}{(2^2-2+1)(2002)(2002+1)} = \frac{1336669}{2005003}. \end{aligned}$$

Perhaps it was wrong to decry the use of Maple earlier: it provides an easy check of the theoretical calculation.

Q1113 You read the following 5 graffiti on an otherwise blank wall:
Exactly 1 of these statements is false.
Exactly 2 of these statements are false.
Exactly 3 of these statements are false.
Exactly 4 of these statements are false.
Exactly 5 of these statements are false.

What can you make of this?

ANS. The statements are mutually inconsistent, so at most one of them is true. If they are all false the last would then have to be false too, which is absurd. Hence exactly one of them is true, that is 'Exactly four of these statements are false'.

That is not the end of the matter. Suppose there had been only one statement, and it read 'This statement is false'. This would be a paradox, but it cannot be examined here.

Q1114 What is the probability that a year chosen at random has 53 Sundays?

ANS. This is another trick question that rather unfortunately found its way into the NSW HSC mathematics examination around 25 years ago. The point is that leap years occur in years divisible by 4 except for some years divisible by 100. The only leap years divisible by 100 are those divisible by 400. Thus 2004, 2180 and 2400 are leap years but 2100 is not. The number of days in 400 consecutive years is $400 \times 365 + 97 = 146097 = 7 \times 20871$. [As this is divisible by 7 there is no need to analyse 2800 consecutive years. The Gregorian calendar repeats itself every 400 years.] Thus in any 400 year interval of time there are $20871 = 400 \times 52 + 71$ Sundays. Thus in that interval 71 out of the 400 years have 53 Sundays. Thus the required probability is 71/400.

OBJECTION: The time this planet takes to go round the sun is not exactly 365.2425 days.

RESPONSE: Well yes, eventually the Gregorian calendar and the above answer will need a minor adjustment.

OBJECTION: The length of the year is not constant over long periods of time.

RESPONSE: No doubt this is so, but further analysis of this matter is too hard.

Q1115 If n is a positive integer, then it is known that $\dfrac{(2002n)!}{(n!)^{2002}}$ is an integer. Determine the highest power of 2002 that divides this number.

ANS. It is worth while noting just why it is that a 'multinomial coefficient' M like that displayed is an integer. These multinomial coefficients can be written as a product of binomial coefficients. Thus, for example:

$$\frac{(n_1+n_2+n_3+n_4)!}{n_1!\cdot n_2!\cdot n_3!\cdot n_4!} = \left[\frac{(n_1+n_2+n_3+n_4)!}{n_1!\cdot(n_2+n_3+n_4)!}\right]\cdot\left[\frac{(n_2+n_3+n_4)!}{n_2!\cdot(n_3+n_4)!}\right]\cdot\left[\frac{(n_3+n_4)!}{n_3!\cdot n_4!}\right].$$

A similar expression could be written out for 2002 symbolic integers $n_1, n_2, \ldots, n_{2002}$ instead of 4. It is then valid to take $n = n_1 = n_2 = n_3 = n_4$ in the simpler example or $n = n_1 = n_2 = \ldots = n_{2002}$ to show that such an expression is an integer. As $2002 = 2\times 7\times 11\times 13$ we now have to find the highest powers of the primes 2, 7, 11 and 13 dividing the given integer $M = (2002n)!(n!)^{-2002}$.

In general, if p is a prime and m is a positive integer, the exponent d of the highest power of p that divides m is given by the formula:

$$\left\lfloor\frac{m}{p}\right\rfloor + \left\lfloor\frac{m}{p^2}\right\rfloor + \left\lfloor\frac{m}{p^3}\right\rfloor + \ldots$$

where for $x > 0$ $\lfloor x\rfloor$ denotes the greatest integer k such that $k \le x$. Thus, given the prime p and the above integer M, the highest power of p dividing M is:

$$d_p = \left\{\left\lfloor\frac{2002n}{p}\right\rfloor - 2002\left\lfloor\frac{n}{p}\right\rfloor\right\} + \left\{\left\lfloor\frac{2002n}{p^2}\right\rfloor - 2002\left\lfloor\frac{n}{p^2}\right\rfloor\right\} + \left\{\left\lfloor\frac{2002n}{p^3}\right\rfloor - 2002\left\lfloor\frac{n}{p^3}\right\rfloor\right\} + \ldots$$

The required answer is now the minimum of d_2, d_7, d_{11}, and d_{13}.

Q1116 A game of draughts has to be abandoned as a draw even though the player to move has two kings and no other pieces while the opponent has only one king and no other pieces. In what positions could this happen?

ANS. The more evident solutions to the question are those situations where the king of the player not to move is on a diagonal adjacent to both kings of the other player and able to capture either. Thus, no matter what move is chosen, it is still possible to capture the king that has not moved. Rather surprisingly there are other solutions in which the three kings are all close to one corner. If the natural 'co-ordinates' are used for the draughts board, and the side to move has kings on the squares 11 and 22 while the other side has a lone king on square 13, no win is possible. You may wish to search for another such situation.

Q1117 Prove that for positive real numbers a, b, c, d, e:

$$\frac{1}{a} + \frac{1}{b} \;\ge\; \frac{4}{a+b}, \tag{1}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} + \frac{64}{e} \;\ge\; \frac{256}{a+b+c+d+e}. \tag{2}$$

ANS. As $(a-b)^2 = (a+b)^2 - 4ab \ge 0$, we deduce $(a+b)^2 \ge 4ab$ and then that $(a+b)(b^{-1}+a^{-1}) \ge 4$ for a, b positive. Divide this last by $(a+b)$ and inequality (1) is obtained.

This inequality may now be used three times over to obtain (2):

$$\begin{aligned}\frac{1}{a}+\frac{1}{b}+\frac{4}{c}+\frac{16}{d}+\frac{64}{e} &\geq \frac{4}{a+b}+\frac{4}{c}+\frac{16}{d}+\frac{64}{e}\\ &\geq \frac{16}{a+b+c}+\frac{16}{d}+\frac{64}{e}\\ &\geq \frac{64}{a+b+c+d}+\frac{64}{e}\\ &\geq \frac{256}{a+b+c+d+e}.\end{aligned}$$

Q1118 Show that any polygon may be dissected into acute angled triangles.

ANS. The problem immediately reduces to showing that any triangle may be dissected into finitely many acute angled triangles. Yet this special case seems quite tricky. We begin with a triangle whose angles are $108°$, $36°$ and $36°$.

Take a regular pentagon with vertices (in order) A, B, C, D and E. Let Z denote the centre of the pentagon. Let AB and DC intersect in X and let AE and CD intersect in Y. Then AXY has angles as above and may be dissected into 7 acute angled triangles. These are ZAB, ZBC, ZCD, ZDE, ZEA, XCB and YED.

You may now try to convince yourself that a distorted version of the above strategy works for any triangle AXY with an obtuse angle at A.

Q1119 Let p, q be two positive integers with no factors in common (except 1) and $p < q$. It is desired to express p/q in the 'Egyptian fraction' form:

$$\frac{1}{n_1}+\frac{1}{n_2}+\ldots\frac{1}{n_r}$$

with $n_1 < n_2 < \ldots < n_r$ for some $r \geq 1$. Show that this is always possible.

ANS. Perhaps surprisingly, this may be handled by an induction on the numerator p working with all $q > p$ at once. The statement is clear when $p = 1$. Thus it is sufficient to prove it for the fraction P/Q (here $P > 1$ and $Q > P$ are coprime integers) under the inductive hypothesis that it is true for all p/q with $p < P$. Let N be the least integer greater than Q/P. Then $P/Q - 1/N = (NP - Q)/(QN)$ has numerator $p = (NP - Q) < P$ and so the induction proceeds.

[If the above proof seems too abstract, start with an example like $9/22$.]

REMARK: It is now easy to see that the number r of reciprocals of integers needed in this 'Egyptian fraction' expansion of p/q is at most p.

Q1120 (a) Five identical solid cylinders whose diameters are 10 times their height are given. Show how to place them so that each touches each of the others. Can this be done with six such cylinders?

(b) Six identical solid cylinders whose heights are 10 times their diameter are given. Show how to place them so that each touches each of the others. Can this be done with seven such cylinders?

ANS. This is based on part of the November 1957 *Mathematical Games* column in *Scientific American*. Thus in part (b) Martin Gardner had in mind placing three of the identical long cylinders flat on a table top so that there was mutual contact and then placing another three on top of the first three to achieve the desired effect. The intended solution for six short cylinders (identical coins will do) is a bit different. Your problem editor has some recollection of seeing in one of Gardner's many books a solution to part (b) but with seven identical cylinders instead of six. Can anyone find the reference?