

Rearrangement Inequalities

David Angell¹

Let's begin with a puzzle.

x	1	2	3	
y				
xy				total =

You are to write the numbers 4, 5, 6 in the row marked “ y ”; multiply the two numbers in each column to obtain three products xy in the third row; add these products to obtain a total. The problem is to choose the *order* in which 4, 5, 6 are placed so as to obtain the largest possible total. Once you have done this, find the order which will give the *smallest* total.

Of course this is really a very easy puzzle. There are only six possible ways to write 4, 5 and 6 in the three slots; all you have to do is calculate the six totals and then pick out the largest and the smallest. You should have found that the maximum total of 32 is obtained by writing the numbers in increasing order, 4, 5, 6; while the minimum total, 28, is obtained by writing the numbers in decreasing order, 6, 5, 4.

This is an example of a *rearrangement inequality*. In order to write down a precise statement of this result, let's first note that if we had written the x values in a different order, then we should have had to change the order of the y s correspondingly. In fact, it clearly makes no difference in what order we add the products to get the final total; so we may as well assume that the x s appear in increasing order.

Theorem *Rearrangement inequalities for sums of products.* Let x_1, x_2, \dots, x_n be positive real numbers in increasing order, that is, $0 < x_1 \leq x_2 \leq \dots \leq x_n$, and let y_1, y_2, \dots, y_n be positive real numbers. Let z_1, z_2, \dots, z_n be a rearrangement of the y_j , and consider the sum

$$x_1 z_1 + x_2 z_2 + \dots + x_n z_n . \tag{1}$$

This sum has its maximum value when the z_j are arranged in increasing order, and its minimum value when they are arranged in decreasing order.

Comment Observe carefully that we have not said that the maximum occurs *only* when the z_j are in increasing order. If some of the x_j are equal then there may be a different order of z_j for which the sum is still maximal. We are only saying that, compared with the increasing order, there is no other order which actually gives a *greater* sum.

¹David Angell is an Associate Lecturer at UNSW

Examples We shall give some examples first and prove the theorem later.

1. Consider two positive real numbers a and b , with $a \leq b$; take $n = 2$ and $x_1 = a, x_2 = b$ and $\{y_1, y_2\} = \{a, b\}$. Then

$$ab + ba \leq x_1z_1 + x_2z_2 \leq a^2 + b^2 ;$$

ignoring the middle term and dividing both sides by 2 yields

$$ab \leq \frac{a^2 + b^2}{2} ,$$

which is a version of the arithmetic mean–geometric mean inequality.

2. If a, b, c are positive then

$$\frac{a+b}{c} + \frac{c+a}{b} + \frac{b+c}{a} \geq 6 .$$

Proof By symmetry we may assume that $a \leq b \leq c$; take

$$x_1 = a, x_2 = b, x_3 = c \quad \text{and} \quad y_1 = \frac{1}{a}, y_2 = \frac{1}{b}, y_3 = \frac{1}{c} .$$

Then $y_3 \leq y_2 \leq y_1$, and so for any rearrangement z_1, z_2, z_3 of y_1, y_2, y_3 we have

$$x_1z_1 + x_2z_2 + x_3z_3 \geq x_1y_1 + x_2y_2 + x_3y_3 = \frac{a}{a} + \frac{b}{b} + \frac{c}{c} = 3 .$$

Making two particular choices for z_1, z_2, z_3 gives

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 \quad \text{and} \quad \frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq 3 ;$$

adding these and collecting terms with the same denominator proves the result claimed.

3. Let a_1, a_2, \dots, a_n be real numbers in increasing order; take

$$x_1 = a_1^2, \dots, x_n = a_n^2 \quad \text{and} \quad y_1 = \frac{1}{a_1}, \dots, y_n = \frac{1}{a_n} .$$

Then (using only one half of the rearrangement inequality)

$$x_1z_1 + \dots + x_nz_n \geq x_1y_1 + \dots + x_ny_n = a_1 + \dots + a_n .$$

In particular, if we take $n = 2002$ and $z_1 = y_2, z_2 = y_3, \dots, z_{2001} = y_{2002}, z_{2002} = y_1$ we obtain

$$\frac{a_1^2}{a_2} + \frac{a_2^2}{a_3} + \dots + \frac{a_{2001}^2}{a_{2002}} + \frac{a_{2002}^2}{a_1} \geq a_1 + a_2 + \dots + a_{2001} + a_{2002} ,$$

which solves question 6 on the 2002 UNSW mathematics competition.

Instead of multiplying pairs of numbers and adding the products we could have done the reverse: added pairs of numbers and then multiplied the sums. A bit of experimenting (perhaps with $x = 1, 2, 3$ and $y = 4, 5, 6$) suggests that the result is, in a sense, the opposite of the one we have seen above.

Theorem Rearrangement inequalities for products of sums. Let x_1, x_2, \dots, x_n be positive real numbers in increasing order, that is, $0 < x_1 \leq x_2 \leq \dots \leq x_n$, and let y_1, y_2, \dots, y_n be positive real numbers. Let z_1, z_2, \dots, z_n be a rearrangement of the y_j , and consider the product

$$(x_1 + z_1)(x_2 + z_2) \cdots (x_n + z_n) .$$

This product has its maximum value when the z_j are arranged in *decreasing* order, and its minimum value when they are arranged in *increasing* order.

Examples

1. Let $x_1 = 1, x_2 = 2$ and $x_3 = 3$. If z_1, z_2 and z_3 are 4, 5 and 6 (not necessarily in that order), then

$$(1 + 4)(2 + 5)(3 + 6) \leq (x_1 + z_1)(x_2 + z_2)(x_3 + z_3) \leq (1 + 6)(2 + 5)(3 + 4) ,$$

that is,

$$315 \leq (1 + z_1)(2 + z_2)(3 + z_3) \leq 343 .$$

2. Let $x_1 = 1^2, x_2 = 2^2, \dots, x_n = n^2$. If z_1, z_2, \dots, z_n is any arrangement of $1, 2, \dots, n$ then

$$(x_1 + z_1)(x_2 + z_2) \cdots (x_n + z_n) \geq (x_1 + 1)(x_2 + 2) \cdots (x_n + n) .$$

That is,

$$\begin{aligned} (1^2 + z_1)(2^2 + z_2) \cdots (n^2 + z_n) &\geq (1^2 + 1)(2^2 + 2) \cdots (n^2 + n) \\ &= (1 \times 2) \times (2 \times 3) \cdots n(n + 1) \\ &= (n + 1)(n!)^2 . \end{aligned}$$

Another way in which the above ideas can be varied is to consider sums of products of three or more numbers. The maximum result generalises in a fairly obvious way, but, curiously, the minimum result does not. For products of sums it is the minimum which generalises and the maximum which does not.

Theorem Extended rearrangement inequalities. Let x_1, x_2, \dots, x_n be positive numbers in increasing order; let $y_{11}, y_{12}, \dots, y_{1n}$ and $y_{21}, y_{22}, \dots, y_{2n}$ and so on be m collections of n positive real numbers. Let $z_{11}, z_{12}, \dots, z_{1n}$ be an arrangement of $y_{11}, y_{12}, \dots, y_{1n}$, let $z_{21}, z_{22}, \dots, z_{2n}$ be an arrangement of $y_{21}, y_{22}, \dots, y_{2n}$ and so forth. Then the sum of products

$$(x_1 z_{11} z_{21} \cdots z_{m1}) + (x_2 z_{12} z_{22} \cdots z_{m2}) + \cdots + (x_n z_{1n} z_{2n} \cdots z_{mn})$$

takes its maximum value when $z_{j1}, z_{j2}, \dots, z_{jn}$ are in increasing order for each j . The product of sums

$$(x_1 + z_{11} + z_{21} \cdots + z_{m1})(x_2 + z_{12} + z_{22} \cdots + z_{m2}) \cdots (x_n + z_{1n} + z_{2n} \cdots + z_{mn})$$

has its *minimum* value when $z_{j1}, z_{j2}, \dots, z_{jn}$ are in increasing order for each j .

Examples

1. Suppose that $x_1 = 1, x_2 = 2, x_3 = 3$ and $\{y_{11}, y_{12}, y_{13}\} = \{4, 5, 6\}$ and $\{y_{21}, y_{22}, y_{23}\} = \{1, 3, 5\}$. The maximum value of

$$(x_1 z_{11} z_{21}) + (x_2 z_{12} z_{22}) + (x_3 z_{13} z_{23})$$

is

$$(1 \times 4 \times 1) + (2 \times 5 \times 3) + (3 \times 6 \times 5) = 124 ,$$

where the 4, 5, 6 and the 1, 3, 5 have been listed in increasing order. However if we list them both in decreasing order, 6, 5, 4 and 5, 3, 1, we obtain

$$(1 \times 6 \times 5) + (2 \times 5 \times 3) + (3 \times 4 \times 1) = 72 ,$$

which is *not* the minimum value of the sum! In similar fashion, the minimum value of

$$(x_1 + z_{11} + z_{21})(x_2 + z_{12} + z_{22})(x_3 + z_{13} + z_{23})$$

is

$$(1 + 4 + 1)(2 + 5 + 3)(3 + 6 + 5) = 840 ,$$

but the maximum value is *not*

$$(1 + 6 + 5)(2 + 5 + 3)(3 + 4 + 1) = 960 .$$

Exercise

What is the minimum value of the sum of products? What is the maximum value of the product of sums?

2. Let $\theta = \pi/2n$ and consider the product of sums

$$P = (\sin \theta + \cos \theta + \cot \theta)(\sin 2\theta + \cos 2\theta + \cot 2\theta) \cdots \\ \cdots (\sin(n-1)\theta + \cos(n-1)\theta + \cot(n-1)\theta) .$$

Note first of all that

$$0 < \theta < 2\theta < \cdots < (n-1)\theta < \frac{1}{2}\pi ,$$

and that on the interval $0 < x < \frac{1}{2}\pi$ the sine function is increasing, while the cosine and cotangent functions are decreasing.

If we take $x_j = \sin j\theta$ then the x s are in increasing order, so to obtain the minimum product of sums we must arrange $y_{1j} = \cos j\theta$ and $y_{2j} = \cot j\theta$ in increasing order too. Therefore

$$P \geq (\sin \theta + \cos(n-1)\theta + \cot(n-1)\theta) \\ (\sin 2\theta + \cos(n-2)\theta + \cot(n-2)\theta) \cdots \\ \cdots (\sin(n-1)\theta + \cos \theta + \cot \theta) .$$

Since $(n - k)\theta = \frac{1}{2}\pi - k\theta$ we have

$$\cos(n - j)\theta = \sin j\theta \quad \text{and} \quad \cot(n - j)\theta = \tan j\theta ,$$

so this inequality can be rewritten as

$$P \geq (2 \sin \theta + \tan \theta) (2 \sin 2\theta + \tan 2\theta) \cdots \\ \cdots (2 \sin(n - 1)\theta + \tan(n - 1)\theta) .$$

Proof of the rearrangement inequalities for sums of products

We shall show that if the z_j are *not* in increasing order, then there is an arrangement with fewer terms out of order, and for which the sum (1) is the same or larger. If the new arrangement still has terms which are not in increasing order we can repeat the process, again increasing the sum; eventually we shall reach a sequence which has no terms out of order. This will show that the sequence with all terms in increasing order gives the largest possible sum.

Suppose, then, that there is some k for which $z_k > z_{k+1}$. Exchange z_k and z_{k+1} to get a new arrangement z'_1, z'_2, \dots, z'_n . That is,

$$z'_k = z_{k+1} , \quad z'_{k+1} = z_k \quad \text{and} \quad z'_j = z_j \quad \text{if } j \neq k, k + 1 .$$

For convenience we write

$$S' = x_1 z'_1 + x_2 z'_2 + \cdots + x_n z'_n \quad \text{and} \quad S = x_1 z_1 + x_2 z_2 + \cdots + x_n z_n ;$$

most of the products in these two sums are the same, and if we subtract them we get

$$\begin{aligned} S' - S &= (x_k z'_k + x_{k+1} z'_{k+1}) - (x_k z_k + x_{k+1} z_{k+1}) \\ &= x_k z_{k+1} + x_{k+1} z_k - x_k z_k - x_{k+1} z_{k+1} \\ &= (x_{k+1} - x_k)(z_k - z_{k+1}) \\ &\geq 0 , \end{aligned}$$

because each of the bracketed terms is non-negative. Thus $S' \geq S$, and z'_1, z'_2, \dots, z'_n has fewer terms out of order than z_1, z_2, \dots, z_n since z_k and z_{k+1} have been "corrected". As explained above, this proves the maximum inequality.

The result that the sum is minimal when the z_j are in decreasing order may be proved by a very similar argument (*exercise!*), or may be deduced from what we have proved in the following way. Choose a real number M larger than all the y_j , and apply the maximum result to the numbers $M - y_1, M - y_2, \dots, M - y_n$. Each z_j must be replaced by $M - z_j$, and so the sum (1) is

$$\begin{aligned} x_1(M - z_1) + x_2(M - z_2) + \cdots + x_n(M - z_n) \\ = (x_1 + x_2 + \cdots + x_n)M - (x_1 z_1 + x_2 z_2 + \cdots + x_n z_n) . \end{aligned}$$

Now the first sum on the right hand side is fixed, and so the minimum value of $x_1 z_1 + x_2 z_2 + \cdots + x_n z_n$ corresponds to the maximum value of the left hand side. From what we have just proved this occurs when $M - z_1, M - z_2, \dots, M - z_n$ are in increasing order, that is, when z_1, z_2, \dots, z_n are in decreasing order.