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The Legacy of Pythagoras' Theorem

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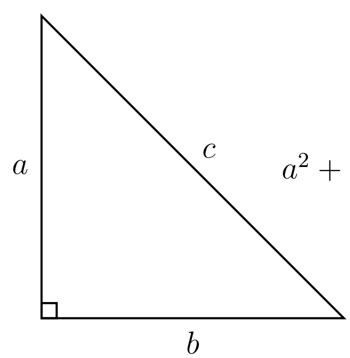
When asked what mathematical result they remember from High School, the average person would probably reply with Pythagoras' Theorem.

In any right triangle, the sum of the squares of the two shorter sides equals the square of the hypotenuse.



$$a^2 + b^2 = c^2$$

Indeed Pythagoras' theorem is memorable not only for its simplicity, but also its unexpectedness. There is no *apriori* reason for there to be such a simple relationship between the sides of a right triangle, and why should the relationship be in terms of squares, why not cubes or some other power?

There are many proofs of Pythagoras' theorem, but let me remind you of one of the simplest.

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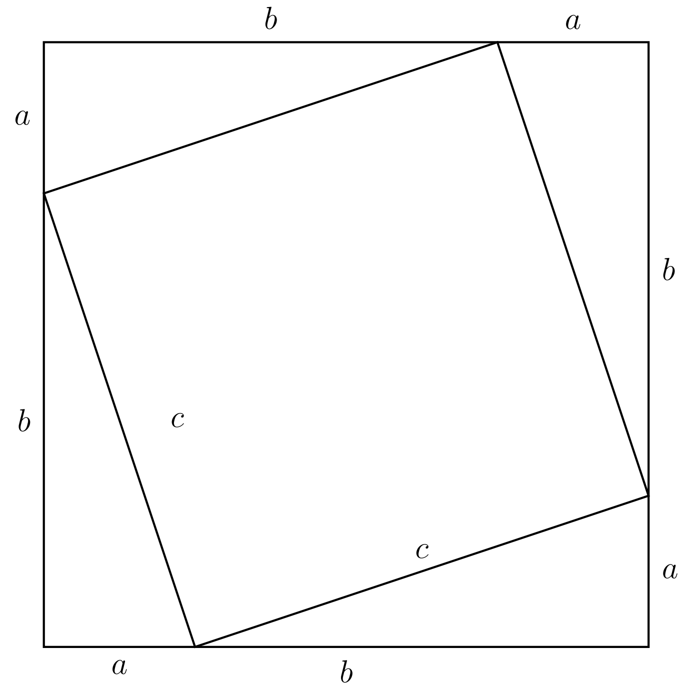
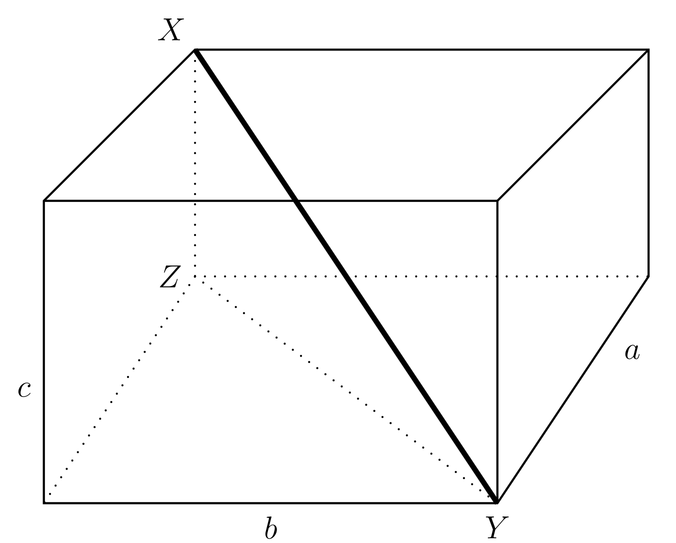


We can arrange 4 triangles around a square of side length c to form a larger square of side length $a+b$. Working out the area of the larger square in two different ways gives

$$(a+b)^2 = c^2 + 4 \times \frac{1}{2}ab \Rightarrow a^2 + b^2 = c^2.$$

Pythagoras' theorem has a number of generalisations.

You can interpret the Pythagorean theorem as a formula for the diagonal of a rectangle with sides a and b. If we then move into three dimensions, the natural question is to find the length of the diagonal of a rectangular prism, sides a, b and c.



The length YZ is clearly equal to $\sqrt{a^2+b^2}$ and $ZY^2+c^2=XY^2$ giving

$$XY=\sqrt{a^2+b^2+c^2},$$

which is the required diagonal. This simple idea can be generalised into higher dimensions.

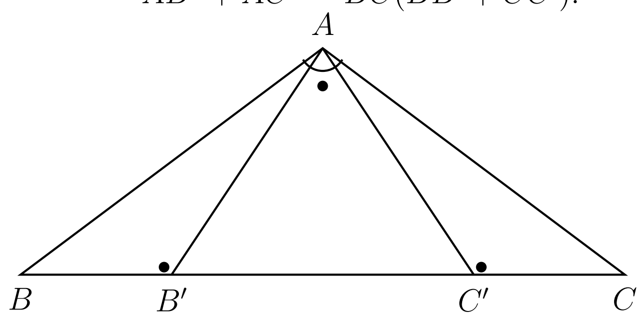
Algebraic generalisations from $a^2+b^2=c^2$ to $a^n+b^n=c^n$ for $n\geq 3$ and a,b,c positive integers, led to the problem known as Fermat's Last Theorem, on which much has recently been written.

The subject of this short article is some **geometric** generalisations of Pythagoras' theorem, which are perhaps not as well known as they should be.

The first of these goes back to the 9th century and is known as *Thabit's Rule*. Thabit ibn Qurra was born in Turkey and was brought to Baghdad in about 870. He wrote books on early algebra (such as it was) and geometry in which appears the following result.

Let ABC be a triangle with an **obtuse** angle at A. Suppose we draw lines AB' and AC' from vertex A meeting BC at B' and C' respectively, such that $\angle AB'B$ and $\angle AC'C$ are equal to the angle at BAC, then

$$AB^2+AC^2=BC(BB'+CC').$$



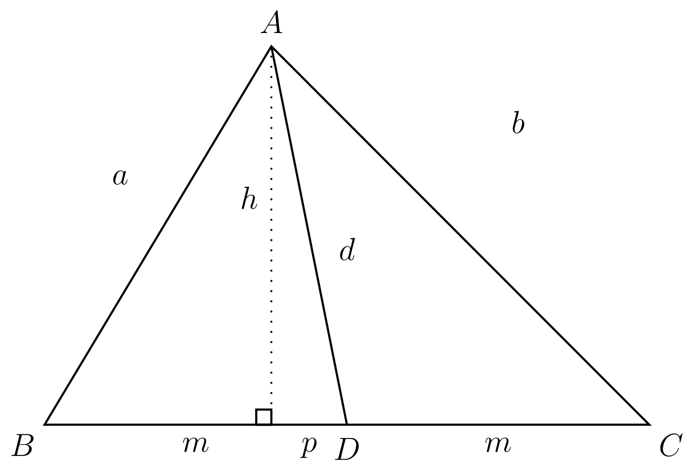
To see this we note that triangle ABC is similar to triangles ABB' and ACC'. Hence

$$\frac{AB}{BB'}=\frac{BC}{AB} \quad \text{and} \quad \frac{AC}{CC'}=\frac{BC}{AC}.$$

Cross multiplying and adding we obtain the desired result. This is a generalisation of Pythagoras' theorem, since if $\angle A=90°$, then B' and C' are the same point and we have Pythagoras' formula.

Another generalisation, which goes back to the ancient Greek geometer Apollonius (who wrote the first treatise on the Conic sections) is *Apollonius' Theorem* which states that:

In ΔABC, if D is the midpoint of BC and $AD=d$, $BD=DC=m$, then $a^2+b^2=2(d^2+m^2)$.



In words it says, in any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median.

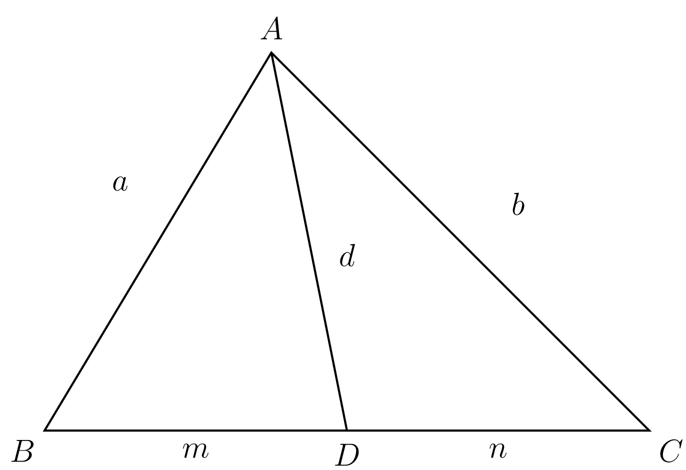
To prove this result, drop a perpendicular from A to BC. Then

$$a^2 = h^2 + (m-p)^2 \quad \text{and} \quad b^2 = h^2 + (m+p)^2$$

by Pythagoras' theorem. Now add to obtain $a^2 + b^2 = 2(h^2 + p^2) + 2m^2$. But $h^2 + p^2 = d^2$ and the result follows.

If $\angle A$ is a rightangle then $d = m = \frac{1}{2}BC$ and this again results in Pythagoras' formula.

This further generalises to a result known as *Stewart's Theorem*, which is a nice application of the cosine rule. Matthew Stewart published this result in 1745.
In ΔABC, suppose that AD divides BC in the ratio $m : n$ (so that $m + n = c$), and $AD = d$, then $a^2n + b^2m = c(d^2 + mn)$. Note that if $m = n$ we have Apollonius' Theorem.



One can give a geometric proof using the same idea as shown above. More easily, using the cosine rule, we have

$$a^2 = d^2 + m^2 - 2dm\cos\angle ADB$$

$$b^2 = d^2 + n^2 - 2dn\cos\angle ADC = d^2 + n^2 + 2dn\cos\angle ADB$$

using $\cos(180 - \theta) = -\cos\theta$. Hence

$$na^2 + mb^2 = d^2(n + m) + mn(m + n) = c(d^2 + mn).$$

Here is a simple exercise for you to try.

Suppose that the line DA in the diagram above bisects the angle at A.
Prove that $d^2 = ab - mn$, using Stewart's theorem.

These results are just a small sample of the many ways in which the famous theorem of Pythagoras has been generalised throughout the two and half thousand years (perhaps much longer) of its history.