Parabola Volume 39, Issue 1 (2003)

Solutions to Problems 1121–1130

Q1121 Prove that $g(x) = 30x^3 + 80x^2 + 72x + 66$ is irreducible over the integers by showing it has no roots modulo 13. What are the roots of $g(x)$ modulo 7?

ANS. The challenge is to show that the equation $g(x) = 0$ has no rational roots. It is strategic to set $f(y) = 900g(y/30) = y^3 + 80y^2 + 2160y + 59400$ and to show instead that the equation $f(y) = 0$ has no rational roots. This f is monic in the sense that the coefficient of the highest power of y is 1. A general theorem states that rational roots of monic polynomials must be integers. One may check that $f(0), f(1), \ldots, f(12)$ are not divisible by 13. Hence for any integer z the integer $f(z)$ is not divisible by 13. Hence there are no integer roots of $f(y) = 0$.
Modulo 7 we have $g(x) = 2x^3 + 3x^2 + 2x + 3$ which has the root $-\frac{3}{2}$, or, as we are working modulo 7, it has the root 2.

Q1122 Given two full 10-litre flasks of mercury, an empty 5-litre flask and an empty 4-litre flask but no other container capable of holding mercury, how can you put 3 litres of mercury in each of the smaller flasks leaving a total of 14 litres in the larger flasks?

ANS. We consider possible configurations with a litres in the 4-litre flask, b litres in the 5 litre flash and c, d litres in the 10 litre flasks. Here we may as well take $c \leq d$. It turns out that there are 90 solutions in integers of $0 \leq a \leq 4,\ 0 \leq b \leq 5,\ 0 \leq c \leq d \leq 10,\ a+b+c+d = 20$. The search for a solution consists of listing first all those that can be obtained from $(0, 0, 10, 10)$ by one pouring operation: these are $(a, b, c, d) = (0, 5, 5, 10)$ and $(4, 0, 6, 10)$. Then one must list all those that can be obtained in two pouring operations, and so on. Eventually $(3, 3, 4, 10)$ emerges after 11 operations by the route
$(0, 0, 10, 10) \longrightarrow (0, 5, 5, 10) \longrightarrow (4, 1, 5, 10) \longrightarrow (0, 1, 9, 10) \longrightarrow (4, 1, 6, 9) \longrightarrow (4, 0, 7, 9) \longrightarrow (0, 4, 7, 9) \longrightarrow (4, 4, 3, 9) \longrightarrow (3, 5, 3, 9) \longrightarrow (3, 0, 8, 9) \longrightarrow (3, 5, 4, 8) \longrightarrow (3, 3, 4, 10)$.

Q1123 Check that $\cos 2\theta = 2\cos^2\theta - 1$ and that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. Now show that 1, $\cos 72°$ and $\cos 144°$ are the roots of the cubic equation $4x^3 - 3x = 2x^2 - 1$, and so evaluate $\cos 72°$ as a surd.

ANS. Put $x = \cos\theta$ in $4x^3 - 3x = 2x^2 - 1$ and the equation $\cos 3\theta = \cos 2\theta$ is obtained. This holds for $\theta = 0°,\ \theta = 72°,\ \theta = 144°$ and so $\cos 0°,\ \cos 72°,\ \cos 144°$ are roots of the cubic equation $4x^3 - 2x^2 - 3x + 1 = 0$. As $\cos 0° = 1,\ 0 < \cos 72° < 1,\ -1 < \cos 144° < 0$, these three roots are mutually different and so are the only roots. We may now use the formula for the roots of a quadratic equation to factorise:

$$\begin{aligned}4x^3 - 2x^2 - 3x + 1 &= (x-1)(4x^2 + 2x - 1)\\ &= 4(x-1)(x - \tfrac{1+\sqrt{5}}{4})(x - \tfrac{1-\sqrt{5}}{4})\end{aligned}$$

and so $\quad \cos 72^\circ = (1+\sqrt{5})/4,\ \cos 144^\circ = (1-\sqrt{5})/4.$

Q1124 Use your answer to the above question to show how an angle of 72° may be constructed in the plane using ruler and compasses only.

ANS. One first constructs a right angle at a point A and then a triangle BAC with $AC = 2BA$ and with a right angle at A. By the Pythagoras theorem $BC = \sqrt{5}\,BA$ and so the compasses may be used to find a point D on BC such that $DC = (\sqrt{5}+1)BA$. The line AC may be extended to a point E with $AE = 4BA$ and a circle drawn with centre C and this radius. This will intersect the perpendicular to DC through D at a point F. The triangle FDC is a right-angled triangle with one of its other angles having cosine $(1+\sqrt{5})/4$. But $\cos 72^\circ = (1+\sqrt{5})/4$.

This is undoubtedly less elegant than the method used by Euclid who was very fond of the regular dodecahedron but is good enough here.

Q1125 Use the second formula in Q1122 to obtain a formula for the solutions of the cubic equation $4x^3 - 3k^2x = a$ valid for a range of values of the real constants k and a. What is the range of values for which there are three real roots? Use a pocket calculator to find the three roots of $4x^3 - 2.9x = 0.5$.

ANS. If we set $x = k\cos\theta$ in $4x^3 - 3k^2x = a$ we obtain

$$\cos 3\theta = a/k^3.$$

Provided $-1 \le a/k^3 \le 1$ this will yield values of 3θ and ultimately of x.

$$\begin{aligned} & x = k\cos\left[\tfrac{1}{3}\cos^{-1}(a/k^3)\right]\\ \text{or}\qquad & x = k\cos\left[\tfrac{1}{3}\cos^{-1}(a/k^3) + 120^\circ\right]\\ \text{or}\qquad & x = k\cos\left[\tfrac{1}{3}\cos^{-1}(a/k^3) + 240^\circ\right].\end{aligned}$$

For example, in $4x^3 - 2.9x = 0.5$ we put

$$\begin{aligned}x &= \sqrt{2.9/3}\cos\theta\\ &= 0.983192\cos\theta\end{aligned}$$

and obtain

$$
\begin{array}{lrcccc}
 & 4x^3 - 3x = 0.5/0.983192^3 = 0.526084 & & & & \\
\text{so} & 3\theta = 58.2587^\circ & \text{or} & 178.2587^\circ & \text{or} & 298.2587^\circ \\
 & \theta = 19.4196^\circ & \text{or} & 59.4196^\circ & \text{or} & 99.4196^\circ \\
 & \cos\theta = 0.943109 & \text{or} & 0.508747 & \text{or} & -0.163663 \\
 & x = 0.927257 & \text{or} & 0.500195 & \text{or} & -0.160912
\end{array}
$$

Q1126 Into how many pieces can a disc be cut with 2 straight lines? Ditto 3 straight lines, 4 straight lines, 5 straight lines, 6 straight lines? How can your answer be generalised?

ANS. Perhaps rather surprisingly, this problem may be handled by using the concept of an Euler characteristic. Suppose, in general, that a disc is cut by straight line segments into F regions. Let E denote the number of edges (either line segments or segments of the bounding circle) and let V denote the number of vertices (either intersections of two segments or intersections of a segment and the circle). Then a simple induction on E shows that $V - E + F = 1$. [The case where there are no line segments at all shows that the complete circle may not be counted as an edge.]

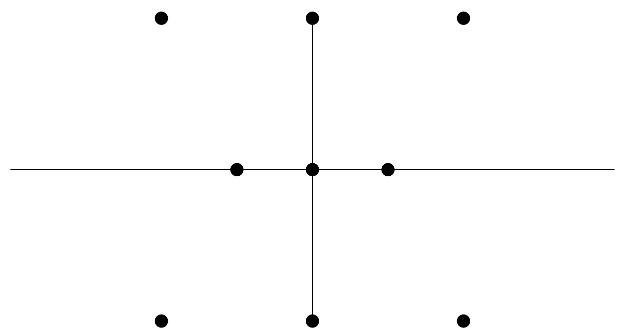
We now dissect the disc with $n > 0$ straight lines and initially suppose that no three lines intersect inside the disc and no two lines intersect on the circle. Let m denote the number of points of intersection inside the circle. Then as each line intersects the circle twice $V = 2n + m$ while $E = 3n + 2m$. Thus $F = n + m + 1$.

The question calls for a formula for the maximum value of F in terms of a fixed n. In general n lines can have at most $n(n-1)/2$ points of intersection, with this maximum being attained if no two of the lines are parallel. So one may draw any n such lines and then draw a circle that contains all the points of intersection. This diagram may then be scaled down to give n lines all intersecting inside the given disc.

Thus the maximum number of regions in the disc that may be obtained by cutting it with n lines is $n + 1 + n(n-1)/2$.

Q1127 Arrange 11 points in the plane so that there are 16 lines each of which contain 3 of these points.

ANS. The topic implicit in this question may well be worth developing further in a future issue of this magazine. Here we solve with the easier question of choosing 9 points in the plane such that there are 10 lines through triples of them. This may be achieved by taking the 9 points: $(-2,2)$, $(0,2)$, $(2,2)$, $(-1,0)$, $(0,0)$, $(1,0)$, $(-2,-2)$, $(0,-2)$ and $(2,-2)$.



With some difficulty a somewhat similar solution for the 11 points problem may be found.

Q1128 [HARD] Here a cyclic quadrilateral Q is a convex quadrilateral whose vertices lie on a circle. There is an ancient formula for the area $|Q|$ of a cyclic quadrilateral with sides of lengths w, x, y, z. Set $s = \frac{1}{2}(w + x + y + z)$. Then $|Q|^2 = (s-w)(s-x)(s-y)(s-z)$. This problem asks you to verify directly the trigonometric identity underlying the formula and thus to prove it. Take the circle containing the vertices of Q to have unit radius and let the four isoceles triangles whose vertices are the centre of the circle and two adjacent vertices of the quadrilateral have angles $2A$, $2B$, $2C$ and $2D$ at their common vertex. Thus $A + B + C + D = 180°$. So here the side lengths may be taken to be $2\sin A$, $2\sin B$, $2\sin C$ and $2\sin D$ and the area of Q is $\frac{1}{2}(\sin 2A + \sin 2B + \sin 2C + \sin 2D)$. You are being asked to prove — or check electronically — that if $A + B + C + D = 180°$ then

$$\begin{aligned} & \frac{1}{4}\left(\sin 2A + \sin 2B + \sin 2C + \sin 2D\right)^2 \\ = \quad & (-\sin A + \sin B + \sin C + \sin D)\cdot(\sin A - \sin B + \sin C + \sin D)\cdot \\ & (\sin A + \sin B - \sin C + \sin D)\cdot(\sin A + \sin B + \sin C - \sin D)\,. \end{aligned}$$

ANS. Although there are geometric proofs of the above formula, it is now possible to check it mechanically. It is best to eliminate D by replacing it with $180° - A - B - C$. Then, for example, the following Maple commands:

```
X:=(sin(2*A)+sin(2*B)+sin(2*C)-sin(2*A+2*B+2*C))^2;
Y:=(-sin(A)+sin(B)+sin(C)+sin(A+B+C))*(sin(A)-sin(B)+sin(C)+sin(A+B+C))
*(sin(A)+sin(B)-sin(C)+sin(A+B+C))*(sin(A)+sin(B)+sin(C)-sin(A+B+C));
simplify(expand(X-4*Y));
quit
```

gives the answer `0` and the formula is established.

Q1129 Show how to deduce the Heron formula for the area of a triangle with sides of lengths a, b, c from the above formula for the area of a cyclic quadrilateral.

ANS. The above formula states that a cycle quadrilateral with sides of lengths a, b, c, d has area

$$\sqrt{\left(\frac{-a+b+c+d}{2}\right) \cdot \left(\frac{a-b+c+d}{2}\right) \cdot \left(\frac{a+b-c+d}{2}\right) \cdot \left(\frac{a+b+c-d}{2}\right)}.$$

Let $d \to 0$ in this formula. In the limit the quadrilateral becomes a triangle with sides of lengths a, b, c and its area is

$$\sqrt{\left(\frac{-a+b+c}{2}\right) \cdot \left(\frac{a-b+c}{2}\right) \cdot \left(\frac{a+b-c}{2}\right) \cdot \left(\frac{a+b+c}{2}\right)}.$$

This is the Heron formula for the area of the triangle.

Q1130 Show that there is a cyclic quadrilateral with sides having different integer lengths in metres such that its area is an integer number of square metres.

ANS. The required cyclic quadrilateral may be chosen so that two opposite angles are right angles and so that one diagonal passes through the centre of the circle through the four vertices. This diagonal may be chosen to have length 65 and be taken to be the common hypotenuse of two right-angled triangles with sides of lengths $39, 52, 65$ and $25, 60, 65$ respectively.