Symmetry and Mathematics

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Question: What do the following objects have in common?

i) The set of complex numbers of modulus one.

ii) An equilateral triangle.

iii) A water molecule.

   Answer: All of these objects can be used to define a mathematical object called a *group*.
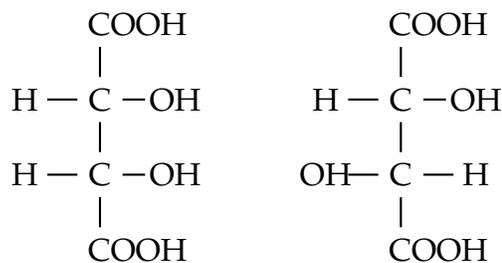
Groups were defined in order to study the concept of symmetry. Before I tell you what a group is, I'd like to talk to you a little bit more about symmetry.

Symmetry is of interest in many fields of science:

i) Our bodies contain proteins which are chains of amino acids twisted into a helix. These helices are always twisted in an anti-clockwise direction. The desire for symmetry has led some biologists to ask if there are any life-forms whose proteins are twisted in clockwise helices.


ii) Proteins are made up of organic molecules which, for the most part, are comprised of carbon, oxygen and hydrogen atoms. Some molecules have symmetric arrangements, while others have asymmetric arrangements. For example,

$$
\begin{array}{cc}
\text{COOH} & \text{COOH} \\
| & | \\
\text{H}-\text{C}-\text{OH} & \text{H}-\text{C}-\text{OH} \\
| & | \\
\text{H}-\text{C}-\text{OH} & \text{OH}-\text{C}-\text{H} \\
| & | \\
\text{COOH} & \text{COOH}
\end{array}
$$

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iii) The "big bang" and "steady state" theories of the universe both predict that the universe is very symmetric.

iv) Much of geometry is based upon the symmetry of plane figures. The Ancient Greeks regarded the circle as being a "perfect" figure, since it has an infinite number of axes of symmetry.

v) The puzzle, Rubik's cube.
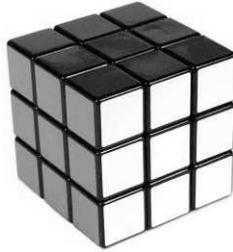
The mechanism of the Rubik's cube has a high degree of symmetry. Some solutions to Rubik's cube are based on this fact. See

http://members.tripod.com/~dogschool/cubegroups.html for further details.

What do we mean by symmetry?

In real life we think of a figure as being "symmetric" if the figure can be reflected along some axis, and the two halves coincide exactly. For example, the butterfly in the following picture is symmetric about a vertical axis through the centre of its thorax.

Some further examples are:

A	is vertically symmetric
E	is horizontally symmetric
H	is both vertically and horizontally symmetric
F	has no symmetry
Z	looks symmetric, but has no symmetry in this sense.

The **Z** example suggests that our intuitive idea of symmetry is not enough to capture what it means for an object to be symmetric.

Mathematicians have taken this intuitive idea of symmetry and made it more precise by providing the following definition:

Definition: A **symmetry** of an object is a transformation of that object which

a) Does not change the appearance of the object as a whole, even though it might move some of its component parts. (Think about operations on a Rubik's cube if you take the stickers off)

b) Leaves distances between corresponding parts of points unchanged.

Properties of Symmetries

1. All objects have at least one symmetry: don't move any points at all! This symmetry is called the **identity transformation** and is denoted by id.

2. Two symmetry transformations can be applied, one after the other, to form another symmetry. (In mathematical language: If s and t are two symmetries of an object, then their product st (formed by transforming the object first by t and then by s) is also a symmetry.

3. Any symmetry transformation s can be "undone". If s is the symmetry transformation, we call the "undoing" transformation s^{-1}. We have

$$ss^{-1} = s^{-1}s = id.$$

Does this remind you of properties of real-valued functions and inverse functions?

Examples:

1. The letter F has only one symmetry, namely the identity transformation.

2. The letter A has two symmetries. These are the identity transformation and a vertical reflection.

Å

3. The symmetries of the letter E are the identity transformation, and a horizontal reflection.
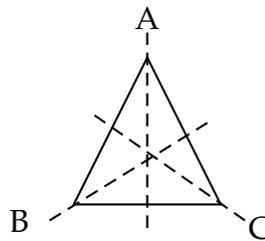
⋯E⋯

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4. The letter H has 4 symmetries: the identity, reflections through horizontal and vertical axes and a rotation through 180 degrees.
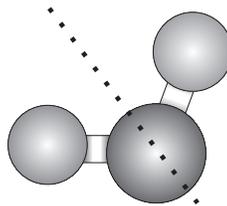

5. The letter Z has two symmetries: the identity and rotation through 180 degrees.

6. An equilateral triangle has 6 symmetries: the identity, rotations through 120 degrees, 240 degrees and a reflection in each of its axes.

7. A water molecule has two symmetries: the identity transformation and a reflection about the axis indicated in the diagram.

8. Symmetries of some objects that appear in advertising and art are discussed on the web page
   `http://www.joma.org/vol-2/framecss/rintel/Math/dihedral.html`[2]

   In a similar vein, why not try to work out the symmetries of the ABC and SBS logos and the Paddlepop Lion?

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[2]Editorial note, February 2014: this is now a dead link

9. Rubik's cube has 43,252,003,274,489,856,000 symmetries if you take the stickers off. By comparison, the universe is less than 630,000,000,000,000,000 seconds old.

10. The set of all complex numbers of modulus 1 forms a circle in the complex plane and therefore has an infinite number of symmetries, all of which correspond to multiplication by a complex number of modulus 1.

So, what is a group?

Mathematicians collect all the symmetries of an object together to form a mathematical object that we call a group. This is the mathematical definition of a group:

Definition: A **group** is a set G of objects, together with an operation $*$ that satisfies the following rules:

1. If s, t are in G, then so is $s*t$ (The product of two symmetries is itself a symmetry).

2. If s, t, u are in G then
$$(s * t) * u = s * (t * u)$$
(it doesn't matter if you calculate $s * t$ first and then multiply the result by u, or multiply s by the product $t * u$).

3. There is a unique identity element id of G (the identity transformation) such that for any g in G we have
$$id * g = g * id = g.$$

4. For every element g of G there is exactly one element g^{-1} of G such that
$$g * g^{-1} = g^{-1} * g = id$$
(g^{-1} "undoes the work of g" - just like an inverse function).

These properties that a set G must satisfy in order to be a group are called the "axioms" (or rules) of a group. Many mathematical objects are defined using lists of axioms. For example, there is a list of axioms (rules satisfied) for the real numbers.

The definition above might look a bit daunting, but these properties exactly mimic the properties we've seen that the set of symmetry transformations of an object possesses. Let's just finish off by looking at a few examples of groups that you've already come across.

1. The set of complex numbers forms a group if we take $*$ to mean addition. Why? To show that this is true, we need to go through and check that this set satisfies each of the axioms for a group, one by one.

Firstly, if z and w are complex numbers, then so is $z + w$, so the first axiom is satisfied. Secondly, if z, w and v are complex numbers, we certainly have $(z + w) + v = z + (w + v)$ and the second axiom is satisfied. The identity element is 0 since, for all complex numbers z we have $z + 0 = 0 + z = z$. Finally, our inverse of z is $-z$, since $z + (-z) = 0 = (-z) + z$. So all the axioms are satisfied and we have a group.

2. Unfortunately, the set of complex numbers does not form a group when we interpret $*$ to mean multiplication. Why not?

   Well, the first three axioms are satisfied. Our identity element is 1. But the fourth axiom is not satisfied. Notice that it says that **every** element of our set must have an inverse. The complex number 0 has no inverse since $0 * z = 0$ for every complex number z. Thus the set of complex numbers does not form a group when $*$ is interpreted as multiplication. Groups have to satisfy **all four** of the axioms listed.

3. We can form groups out of some subsets of the complex numbers when we interpret $*$ as multiplication. If we take the set of all complex numbers and throw away 0, then we have a group. One subset of the complex numbers that comes up frequently in Mathematics is the set of all complex numbers of modulus 1.
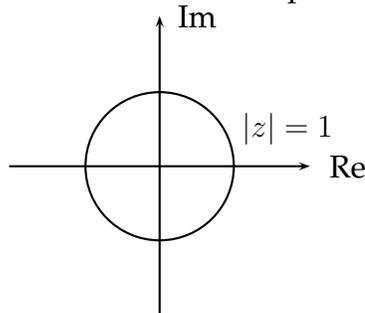
   Remember that this set does not contain 0 since $|0| \neq 1$. Let's check that this set does form a group. Remember $*$ means multiplication.

   Suppose z and w are complex numbers of modulus 1. Then

   $$|z * w| = |z||w| = 1$$

   and so $z * w$ is in our set.

   We've already said that the second axiom is satisfied by all complex numbers, so it will be satisfied here too. The identity element is again 1 which is in our set since $|1| = 1$. So we just have to check whether the inverses are in the set as well. We take an arbitrary complex number of modulus 1 called z. Now

 $$\left| \frac{1}{z} \right| = \frac{1}{|z|} = 1$$

and so $\frac{1}{z}$ is in our set, and

$$z * \frac{1}{z} = 1 = \frac{1}{z} * z.$$

Thus this set of all complex numbers of modulus one satisfies all of the axioms for a group and therefore is a group.

4. The set of all symmetries of a regular polygon also forms a group. These groups are called the **dihedral groups** and have twice as many elements as the number of sides of the polygon.

Let's look at the set of all symmetries of an equilateral triangle:
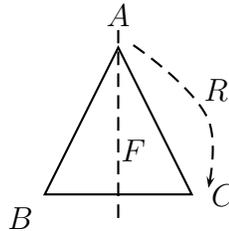
We expect 6 operations since the triangle has three sides. The group here is called D_3: The dihedral group of a regular 3-sided polygon.
The operations are:

|	Operation	Symbol
1.	identity	id
2.	rotate through $120°$	R
3.	rotate through $240°$	R followed by R or R^2
4.	"flip" about an axis through A	F
5.	"flip" about an axis through C	F followed by $R = RF$
6.	"flip" about an axis through B	F followed by $R^2 = R^2F$

where R is a rotation of $120°$ clockwise and F is a "flip" about the vertical axis.

We can see how these operations combine together using a "multiplication table". This is a table that lists all the possible products of pairs of elements.

	id	R	R^2	F	RF	R^2F
id	id	R	R^2	F	RF	R^2F
R	R	R^2	id	RF	R^2F	F
F	F	FR $= R^2F$	FR^2 $= RF$	id	FRF $= R^2F^2$ $= R^2$	FR^2F $= RF^2$ $= R$
RF	RF	RFR $= RR^2F$ $= R$	RFR^2 $= R^2F$	R	$RFRF$ $= RR^2F^2$ $= id$	RFR^2F $= R^2F^2$ $= R^2$
R^2F	R^2F	R^2FR $= RF$	F	R^2	R^2FRF $= R^4F^2$ $= R$	R^2FR^2F $= id$

Let's have a think about how we came up with some of the products in the table. The key ideas are that $F^2 = id$, $R^3 = id$, $FR = R^2F$ and $FR^2 = RF$. What happens if you flip the triangle about an axis through A twice? You get back where you started from. Similarly, R^3 corresponds to rotation through $360°$. Again we end up where we started from. So these two transformations certainly have the same effect as the identity transformation.

Now, why is $FR = R^2F$? If you make yourself an equilateral triangle and do these operations to it, you'll see that you end up in the same place. Both combinations of operations turn our triangle ABC into the triangle BCA as illustrated below:
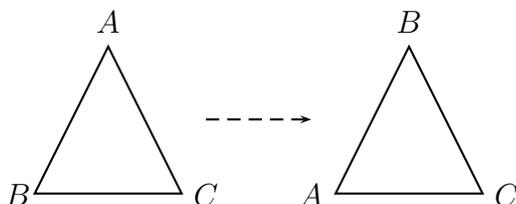
so they correspond to the same symmetry transformation. The same sort of thing happens with FR^2 and RF.

Next, how did we come up with line 5 of the table, for example? Let's do a few of the calculations so that we can get the hang of them. We'll use the ideas that I mentioned above.

Now

$$\begin{aligned}
(R * F) * R &= R * (F * R) \text{ using Axiom 2.} \\
&= R * (R^2 * F) \text{ using the 3rd fact above} \\
&= (R * R^2) * F \text{ using Axiom 2} \\
&= R^3 * F \text{ using Axiom 2} \\
&= id * F \text{ using the 1st fact above} \\
&= F \text{ using Axiom 3}
\end{aligned}$$

and

$$
\begin{aligned}
(RF)R^2 &= R(FR^2) \text{ using Axiom 2.} \\
&= R(RF) \text{ using the 4th fact above} \\
&= R^2F \text{ using Axiom 2.}
\end{aligned}
$$

The other products in this, and the other rows, are found in a similar manner.

You can also find inverses from the table: Just look for two things that multiply together to give the identity transformation. For example, $(R^2)^{-1} = R$.

This article was intended to give you a bit of an idea about what group theory is all about. If you're interested you can find out more on the following web sites:
http://www.members.tripod.com/~dogschool
http://www.joma.org/vol1-2/framecss/rintel/Math/dihedral.html[3]
http://hemsidor.torget.se/users/m/mauritz/math/alg/dihed.htm[4]
http://jwilson.coe.uga.edu/EMT668/EMAT6680.F99/McCallum/WALLPA~1/DEFAULT.HTI

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[3]Editorial note, February 2014: this is now a dead link
[4]Editorial note, February 2014: this is now a dead link