Parabola Volume 39, Issue 2 (2003)

UNSW School Mathematics Competition 2003

Problems and Solutions

Junior Division

1. A cow is inside a square field. Its distances from the nearest three corners are 30m, 40m and 50m. What is its distance to the furthest corner, and what is the size of the field?

   Solution:
   Let s be the side of the square, a be the cow's distance from the nearest edge, and b the cow's distance from the second nearest edge. Then $a^2 + b^2 = 30^2$, $a^2 + (s-b)^2 = 40^2$ and $(s-a)^2 + b^2 = 50^2$. It follows that $(s-a)^2 + (s-b)^2 = 40^2 + 50^2 - 30^2 = 3200$, and the cow is $\sqrt{3200} = 40\sqrt{2}$ metres from the furthest corner. We have $s^2 - 2as = 50^2 - 30^2 = 1600$, $s^2 - 2bs = 40^2 - 30^2 = 700$. Therefore

   $$a = \frac{s^2 - 1600}{2s}, \quad b = \frac{s^2 - 700}{2s}$$

   and

   $$\left(\frac{s^2 - 1600}{2s}\right)^2 + \left(\frac{s^2 - 700}{2s}\right)^2 = 30^2,$$

   or,

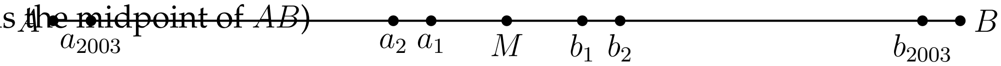
   $$(s^2 - 1600)^2 + (s^2 - 700)^2 = 3600s^2,$$
   $$s^4 - 4100s^2 + 1525000 = 0,$$
   $$s^2 = 2050 \pm 50\sqrt{1071}.$$

   The $+$ sign is appropriate here, and $s = 5\sqrt{82 + 2\sqrt{1071}}$.

2. A regular tetrahedron is to be painted with each face a uniform colour. Four different colours are available, but the same colour may be used on more than one face. How many different patterns are possible? Two patterns are different if the tetrahedra cannot be rotated so that they match. Mirror images are not considered identical.

Solution:
There are 35 colour combinations, and just one of these (four different colours) gives rise to two distinct tetrahedra. So there are 36 different tetrahedra.

3. (M is the midpoint of AB)



2003 Gold and 2003 Silver beans are placed randomly at the 4006 places labelled $a_1, a_2, \cdots, a_{2003}, b_1, b_2, \cdots, b_{2003}$, one bean at each place. Furthermore, $Ma_i = Mb_i$ for $i = 1, 2, \cdots, 2003$.

Show that the total distance of the Gold beans from A is equal to the total distance of the Silver beans from B.

Solution:
See Senior (1).

4. ACD is an equilateral triangle, B lies on AD, closer to A than to D. Let $AC = b$, $AB = c$, $BC = a$.

   (i) Find a relation between a, b, c.

   (ii) Find integer values for a, b, c.

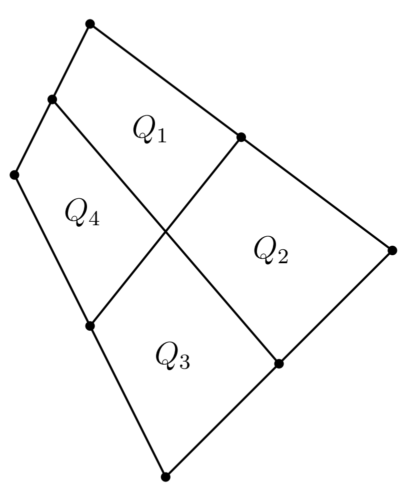
   (iii) Find infinitely many integer values for a, b, c.

Solution:
See Senior (4).

5. Find a positive real number with the property that if we interchange its first and fifth digits after the decimal point then it increases by a factor of 2501.

Solution:
See Senior (5).

6. The convex quadrilateral $ABCD$ is divided into four smaller quadrilaterals by two straight lines joining the midpoints of opposite sides. Denote the areas of the four small quadrilaterals by Q_1, Q_2, Q_3, Q_4 in the order shown. Prove that $Q_1 + Q_3 = Q_2 + Q_4$.

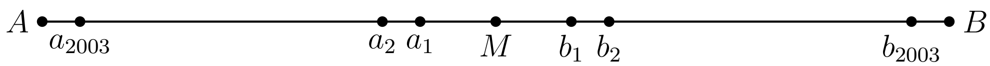


Solution:

(One solution of several.) Join the vertices of the quadrilateral to the centre with four "radii". Each radius divides one of the smaller quadrilaterals into two triangles, so we have eight triangles. The two triangles between each pair of radii have equal area, so there are four pairs of triangles with equal areas. Each of $Q_1 + Q_3$ and $Q_2 + Q_4$ is the sum of the areas of four of the triangles, one from each of the four pairs. So $Q_1 + Q_3 = Q_2 + Q_4$.

Senior Division

1. (M is the midpoint of AB)



   2003 Gold and 2003 Silver beans are placed randomly at the 4006 places labelled $a_1, a_2, \cdots, a_{2003},\ b_1, b_2, \cdots, b_{2003}$, one bean at each place. Furthermore, $Ma_i = Mb_i$ for $i = 1, 2, \cdots, 2003$.

   Show that the total distance of the Gold beans from A is equal to the total distance of the Silver beans from B.

   Solution:
   If a pair of points (a_i, b_i) is occupied by beans of different colours, then they contribute equally to the sums of distances.

   If the pair of points are occupied by say two Gold beans, they contribute D to the sum of distances, where D is the distance from A to B. But the number of (Gold, Gold) pairs is equal to the number of (Silver, Silver) pairs.

   So the sums of distances are equal.

2. The 5–digit numbers $11111,\ 11112,\ \cdots, 99999$ are written on different cards. These cards are shuffled and arranged in a long row so as to form a 444445–digit number. Prove that no matter how the cards are arranged, the 444445–digit number cannot be a power of 2003.

   Solution:
   Let the numbers be arranged in the order $a_1,\ \cdots, a_{88889}$. Then the 444445 digit number is
   $$N = a_1 \times 10^{5\times 88888} + a_2 \times 10^{5\times 88887} + \cdots + a_{88889}.$$

   We will show that N is divisible by 11111, and so, since 2003 is a prime that does not divide 11111, N is not a power of 2003.

   First we note that $10^{5k} \equiv 1 \pmod{11111}$. It follows that
   $$N \equiv a_1 + \cdots + a_{88889} \equiv 11111 + 11112 + \cdots + 99999 \pmod{11111}.$$

   Next we note that
   $$11111 + 11112 + \cdots + 99999 = \frac{11111 + 99999}{2} \times 88889 \equiv 0 = 55555 \times 88889$$

   $\pmod{11111}$.

3. Let $a,\ b,\ c$ be the sides of a triangle. Prove that

$$\frac{3}{2} \le \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} < 2.$$

Solution:
The inequality

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \ge \frac{3}{2}$$

holds for all positive $a,\ b,\ c$. It is equivalent to

$$2a(a+b)(a+c) + 2b(b+a)(b+c) + 2c(c+a)(c+b) \ge 3(a+b)(b+c)(c+a)$$

or,

$$\begin{aligned} &2a^3 + 2b^3 + 2c^3 + 2a^2b + 2ab^2 + 2b^2c + 2bc^2 + 2c^2a + 2ca^2 + 6abc \\ &\ge 3a^2b + 3ab^2 + 3b^2c + 3bc^2 + 3c^2a + 3ca^2 + 6abc, \end{aligned}$$

or,

$$2a^3 + 2b^3 + 2c^3 - a^2b - ab^2 - b^2c - bc^2 - c^2a - ca^2 \ge 0,$$

or,

$$(a+b)(a-b)^2 + (b+c)(b-c)^2 + (c+a)(c-a)^2 \ge 0,$$

which is clearly true. The inequality

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} < 2$$

is equivalent to

$$a(a+b)(a+c) + b(b+a)(b+c) + c(c+a)(c+b) < 2(a+b)(b+c)(c+a),$$

or,

$$\begin{aligned} &a^3 + b^3 + c^3 + a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 3abc \\ &< 2a^2b + 2ab^2 + 2b^2c + 2bc^2 + 2c^2a + 2c^2a + 2ca^2 + 4abc, \end{aligned}$$

or,

$$-a^3 - b^3 - c^3 + a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + abc > 0.$$

But since $a,\ b,\ c$ are the sides of a triangle,

$$a+b-c > 0, \quad b+c-a > 0, \quad c+a-b > 0$$

and

$$(a+b-c)(b+c-a)(c+a-b) > 0,$$

or,

$$-a^3 - b^3 - c^3 + a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 - 2abc > 0.$$

The desired inequality obviously holds.

4. ACD is an equilateral triangle, B lies on AD, closer to A than to D. Let $AC = b$. $AB = c$, $BC = a$.

   (i) Find a relation between $a,\ b,\ c$.

   (ii) Find integer values for $a,\ b,\ c$.

   (iii) Find infinitely many integer values for $a,\ b,\ c$.

   Solution:

   (i) Using Pythagoras' theorem or the cosine rule we find that

   $$a^2 = b^2 - bc + c^2.$$

   (ii) One solution in integers is $a = 7$, $b = 8$, $c = 3$.

   (iii) Infinitely many solutions in integers can be obtained by multiplying any one solution throughout by an arbitrary integer.

   In an attempt to find all solutions in integers, we can write

   $$a^2 = \frac{b^3 + c^3}{b + c}.$$

   We can then let $b = k + l, \quad c = k - l$, to find

   $$a^2 = k^2 + 3l^2,$$

   or,

   $$(a + k)(a - k) = 3l^2.$$

   If we then set $a + k = 3p^2, \quad a - k = q^2,\ l = pq, \quad k = \frac{3p^2 - q^2}{2}$,

   $$a = \frac{3p^2 + q^2}{2}, \quad b = \frac{3p^2 + 2pq - q^2}{2}, \quad c = \frac{3p^2 - 2pq - q^2}{2}$$

   and, say, choose p and q both odd, then we have one doubly infinite set of solutions.

   If we set $a + k = p^2, \quad a - k = 3q^2$, we will find another.

5. Find a positive real number with the property that if we interchange its first and fifth digits after the decimal point then it increases by a factor of 2501.

   Solution:
   Let the number be x. We first show that $x < 1$. Suppose $x \geq 1$, and we interchange the first and fifth decimals. Then the new number is no more than $x + 1$, so is no more than $2x$.

So suppose $x = 0.a000b + y$, where y is a number with 0s in the first and fifth decimal positions. After interchanging a and b we get the number $0.b000a + y$. Thus we have

$$0.b000a + y = 2501(0.a000b + y)$$

or, multiplying by 10^5,

$$10^4 b + a + 10^5 y = 2501(10^4 a + b) + 2501 \times 10^5 y,$$

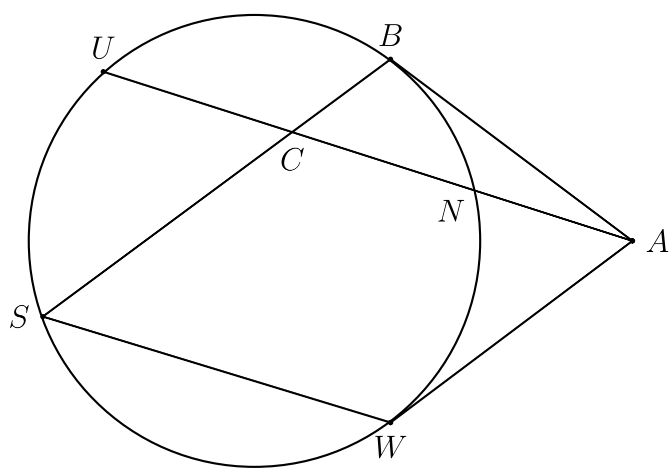
or,

$$7499b = 25009999a + 250000000y.$$

It follows that $a = 0$ and $y = \frac{7499b}{250000000} = \frac{29996b}{1000000000}$. We now try values of b from 1 to 9, and hope to find a value of b that will give a value of y with 0 in its first and fifth decimal digits. For $b = 7$ we find $y = 0.000209972$. So the (unique) solution to our problem is

$$x = 0.00007 + 0.000209972 = 0.000279972.$$

6. AB and AW are tangents to a circle from an external point A. UNA is a secant through A, cutting the circle at U and N. WS is a chord of the circle with $UN || SW$. SB intersects UN at C. Prove that $UC = CN$.



Solution:
Let O be the centre of the circle. Join O to A, B, C and W. Let $\angle BSW = \alpha$. Then

$\angle BCA = \alpha$ (corresponding angles). Also, $\angle BOW = 2\alpha$ (angle at centre = twice angle at circumference standing on same arc). Thus (by symmetry), $\angle BOA = \alpha$.

Since $\angle BCA = \angle BOA$, $BCOA$ is a cyclic quadrilateral. It follows that $\angle OCA = \angle OBA = 90^{\circ}$. The rest is easy!