Parabola Volume 39, Issue 2 (2003)

Solutions to Problems 1131–1140

Q1131 Suppose A and B are two equally strong tennis players. Is it more probable that A will beat B in three sets out of 4 or in 5 sets out of eight?

ANS. If A and B play n sets, the probability that A will win some specified r sets (for example, the first r sets) and lose the remaining $n-r$ is $\left(\frac{1}{2}\right)^n$. (His probability of winning one set being $\frac{1}{2}$, his probability of winning the first r sets $\left(\frac{1}{2}\right)^r$, his probability of losing the last $(n-r)$ sets is $\left(1-\frac{1}{2}\right)^{n-r} = \left(\frac{1}{2}\right)^{n-r}$, and multiplying these gives the stated result.) Hence the probability of A beating B in exactly r sets out of n is ${}^nC_r\left(\frac{1}{2}\right)^n$ (where ${}^nC_r$ is the number of different ways one can choose r objects from n given objects - in this case sets of tennis). Hence A's probability of winning exactly 3 sets out of 4 is ${}^4C_3\left(\frac{1}{2}\right)^4 = 4.\frac{1}{2^4} = \frac{1}{4}$ whilst his probability of winning exactly 5 sets out of 8 is

$$\begin{aligned} {}^8C_5\left(\frac{1}{2}\right)^8 &= \frac{8.7.6.5.4}{5.4.3.2.1}\cdot\frac{1}{2^8} \\ &= \frac{7}{32}, \text{ a somewhat smaller expectation.} \end{aligned}$$

Of course, A's probability of winning "at least" 5 sets out of 8 is considerably larger than his probability of winning at least 3 out of 4 (the probabilities are $\frac{93}{256}$ and $\frac{5}{16}\left(=\frac{80}{256}\right)$ respectively).

Q1132 Triball is a three player game. In each round, the winner scores a point, the runner up is awarded b points, and the loser gets c points, where $a > b > c$ are positive integers. One day Xavier, Yvonne and Zachary played some triball and the final score was

$$\text{Xavier} - 20 \qquad \text{Yvonne} - 10 \qquad \text{Zachary} - 9.$$

Yvonne won the second round. Who won the first round, and how many points did Zachary get in the last round?

ANS. Since the total number of points scored is 39, there must have been 3 rounds played with $a+b+c = 13$. (Note $a+b+c > 3$) Yvonne's score, including an a, totalled fewer than 13 points. She **must** have come third in the other two rounds, and $a+2c = 10$. We deduce that $b = c+3$. Since Zachary's average score per round exceeds

6, we must have $a \geq 7$ $2c \leq 3$ yielding $c = 1$ as the only possibility (since c is a **positive** integer). Therefore $a = 8,\ b = 4,\ c = 1$. Xavier must have won the 1st and 3rd rounds and come second in the 2nd round. Hence Zachary must have come third in the 2nd round, and second in the first round and also the last round, in which he scored 4 points.

Q1133 Let us assume that in a knockout tennis tournament the players always play true to form, so that the stronger player invariably defeats a weaker opponent. If there are 16 entrants in the tournament, and the draw is decided by lot, what is the probability that the final (in round 5) is played between the strongest two players?

ANS. Whichever of the 16 places in the draw is allotted to the strongest player A, the remaining 15 places are equally likely for the second strongest B. Of these, B is in the opposite half of the draw to A, and 7 in the same half. Thus the probability that A and B are in the final is $\frac{8}{15}$.

Q1134 Let $a_1, a_2, \ldots, a_n$ be positive numbers. Prove that

$$\frac{a_2 + a_3 + \cdots + a_n}{a_1} + \frac{a_1 + a_3 + \cdots + a_n}{a_2} + \cdots$$
$$\cdots + \frac{a_1 + a_2 + \cdots + a_{n-1}}{a_n} \geq n^2 - n\ ,$$

and deduce that

$$\bigl(a_1 + a_2 + \cdots + a_n\bigr)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2\ .$$

ANS. We may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. Take

$$x_1 = \frac{1}{a_1},\ x_2 = \frac{1}{a_2},\ \ldots,\ x_n = \frac{1}{a_n}$$
$$y_1 = a_1,\ \ y_2 = a_2,\ \ldots,\ y_n = a_n.$$

Now $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$, so if $z_1, z_2, \ldots, z_n$ is any rearrangement of $y_1, y_2, \ldots, y_n$ we have

$$\begin{aligned} x_1 z_1 + x_2 z_2 + \cdots + x_n z_n &\geq x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \\ &= 1 + 1 + \cdots + 1 \\ &= n. \end{aligned}$$

Choosing $z_1 = a_2,\ z_2 = a_3, \ldots, z_{n-1} = a_n,\ z_n = 1$, we have

$$\frac{a_2}{a_1} + \frac{a_3}{a_3} + \cdots + \frac{a_n}{a_{n-1}} + \frac{a_1}{a_n} \geq n.$$

Similarly

$$\frac{a_3}{a_1} + \frac{a_4}{a_2} + \cdots + \frac{a_1}{a_{n-1}} + \frac{a_2}{a_n} \geq n$$
$$\vdots$$
$$\frac{a_n}{a_1} + \frac{a_1}{a_2} + \cdots + \frac{a_{n-2}}{a_{n-1}} + \frac{a_{n-1}}{a_n} \geq n\ ;$$

adding these inequalities gives

$$\frac{a_2 + a_3 + \cdots + a_n}{a_1} + \frac{a_1 + a_3 + \cdots + a_n}{a_2} + \cdots$$
$$\cdots + \frac{a_1 + a_2 + \cdots + a_{n-1}}{a_n} \geq (n-1)n$$

which proves the first inequality. But clearly

$$\frac{a_1}{a_1} + \frac{a_2}{a_2} + \cdots + \frac{a_n}{a_n} = n$$

and adding this to the first inequality gives the second.

Q1135 Prove the rearrangement inequalities for products of sums.

ANS. Let $x_1, x_2, \ldots, x_n$ be positive real numbers with $x_1 \leq x_2 \leq \cdots \leq x_n$ and let $y_1, y_2, \ldots, y_n$ be positive real numbers. Suppose that $z_1, z_2, \ldots, z_n$ is a rearrangement of $y_1, y_2, \ldots, y_n$ and that the z_j are **not** in increasing order. We shall show that there is an arrangement with fewer terms out of order for which the product

$$(x_1 + z_1)(x_2 + z_2) \cdots (x_n + z_n)$$

is the same or smaller.

Suppose, then, that there is some t for which $z_k > z_{k+1}$. Exchange z_k and z_{k+1} to get a new arrangement $z'_1, z'_2, \ldots, z'_n$. That is,

$$z'_k = z_{k+1},\ z'_{k+1} = z_k \quad \text{and} \quad z'_j = z_j \quad \text{if} \quad j \neq k, k+1.$$

For convenience we write

$$P = (x_1 + z_1)(x_2 + z_2) \cdots (x_n + z_n)$$

and

$$P' = (x_1 + z'_1)(x_2 + z'_2) \cdots (x_n + z'_n);$$

most of the terms in these two products are the same and we have

$$\begin{aligned}\frac{P'}{P} &= \frac{x_k + z'_k}{x_k + z_k}\,\frac{z_{k+1} + z'_{k+1}}{x_{k+1} + z_{k+1}} \\ &= \frac{x_k + z_{k+1}}{x_k + z_k}\,\frac{z_{k+1} + z_k}{x_{k+1} + z_{k+1}} \\ &= \frac{x_k z_{k+1} + x_k z_k + x_{k+1} z_{k+1} + z_k z_{k+1}}{x_k x_{k+1} + x_k z_{k+1} + x_{k+1} z_k + z_k z_{k+1}}. \qquad (1)\end{aligned}$$

But

$$x_k z_k + x_{k+1} z_{k+1} \le x_k z_{k+1} + x_{k+1} z_k;$$

to prove this, either show that $RHS - LHS \ge 0$ or use the rearrangement inequality for sums of products. So in (1), the numerator is less than or equal to the denominator, so $P'/P \le 1$ and $P' \le P$. We can repeat this procedure as often as needed until the z_k are in increasing order; at each stage the product of sums remains the same or decreases. Hence, the minimum value of the product occurs when $z_1, z_2, \ldots, z_n$ are in increasing order.

The proof that the maximum occurs when $z_1, z_2, \ldots, z_n$ are in decreasing order is very similar.

Q1136 Arrange the numbers $1, 2, \ldots, 2n$ into an order $x_1, x_2, \ldots, x_{2n}$ so as to obtain the maximum value of

$$x_1x_2 + x_3x_4 + x_5x_6 + \cdots + x_{2n-1}x_{2n}\ .$$

Also, find the minimum value of the sum.

ANS. The maximum possible sum is

$$1 \times 2 + 3 \times 4 + 5 \times 6 + \cdots + (2n-1)2n$$

and the minimum is

$$1(2n) + 2(2n-1) + 3(2n-2) + \cdots + n(n+1).$$

Proof. To make the explanation easier we shall relabel the numbers as $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ and consider the sum

$$x_1y_1 + x_2y_2 + x_3y_3, \ldots + x_ny_n.$$

We may assume that $x_j < y_j$ for each j (because $x_jy_j = y_jx_j$) and that $x_1 < x_2 < \cdots < x_n$ (reorder the terms if necessary).

For the maximum, if the arrangement of x s and y s is not as shown then the sequence

$$x_1, y_1, x_2, y_2, \ldots, x_n, y_n$$

is not in increasing order, so either $x_k > y_k$ or $y_k > x_{k+1}$ for some k. But the former alternative, by assumption, is never true, so the latter must hold. Let

$$\begin{aligned} x'_{k+1} &= y_k \qquad \text{and} \qquad x'_j = x_j \quad \text{for} \quad j \neq k+1 \\ y'_k &= x_{k+1} \quad \text{and} \qquad y'_j = y_j \quad \text{for} \quad j \neq k \end{aligned}$$

– that is, interchange y_k and x_{k+1} – and write

$$\begin{aligned} S &= x_1y_1 + x_2y_2 + \cdots + x_ny_n \\ S' &= x'_1y'_1 + x'_2y'_2 + \cdots + x'_ny'_n. \end{aligned}$$

Then

$$\begin{aligned} S' - S &= x'_ky'_k + x'_{k+1}y'_{k+1} - x_ky_k - x_{k+1}y_{k+1} \\ &= x_kx_{k+1} + y_ky_{k+1} - x_ky_k - x_{k+1}y_{k+1} \\ &= (y_k - x_{k+1})(y_{k+1} - x_k), \end{aligned}$$

and this is positive since $y_{k+1} > x_{k+1} > x_k$. So if $x_1, x_2, \ldots, y_n$ are not as shown then S does not have its maximum.

To prove the minimum result we show first that $y_1 > y_2 > \cdots > y_n$. If not, then there exist k, ℓ with

$$k < \ell, \quad y_k < y_\ell.$$

Define

$$\begin{gathered} x'_j = x_j \qquad \text{for all } j \\ y'_k = y_\ell,\ y'_\ell = y_k,\ y'_j = y_j \quad \text{if } j \neq k, \ell \end{gathered}$$

– that is, interchange y_k and y_ℓ – and let

$$\begin{aligned} S &= x_1y_1 + x_2y_2 + \cdots + x_ny_n \\ S' &= x'_1y'_1 + x'_2y'_2 + \cdots + x'_ny'_n \end{aligned}$$

as above. Then

$$\begin{aligned} S' - S &= x'_ky'_k + x'_\ell y'_\ell - x_ky_k - x_\ell y_\ell \\ &= x_ky_\ell + x_\ell y_k - x_ky_k - x_\ell y_\ell \\ &= (x_k - x_\ell)(y_\ell - y_k), \end{aligned}$$

which is negative since $x_k < x_\ell$ and $y_k < y_\ell$. So S is not the minimal sum under these conditions; we can only get a minimum when

$$x_1 < x_2 < \cdots < x_n < y_n < \cdots < y_2 < y_1$$

and the sum is as shown.

Comment We can also evaluate both sums. You can prove by induction that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1(2n + 1).$$

Hence the maximum is

$$\begin{aligned}
&1 \times 2 + 3 \times 4 + 5 \times 6 + \cdots + (2n-1)(2n) \\
&\quad = (2^2 - 2) + (4^2 - 4) + (6^2 - 6) + \cdots + ((2n)^2 - 2n) \\
&\quad = 2^2(1^2 + 2^2 + 3^2 + \cdots + n^2) - (2 + 4 + 6 + \cdots + 2n) \\
&\quad = \frac{2}{3}n(n+1)(2n+1) - \frac{1}{2}n(2 + 2n) \\
&\quad = \frac{1}{3}n(n+1)(4n + 2 - 3) \\
&\quad = \frac{1}{3}n(n+1)(4n-1),
\end{aligned}$$

and the minimum

$$\begin{aligned}
&1(2n) + 2(2n-1) + 3(2n-2) + \cdots + n(n+1) \\
&\quad = 1(2n+1-1) + 2(2n+1-2) + 3(2n+1-3) + \cdots + n(2n+1-n) \\
&\quad = (1 + 2 + 3 + \cdots + n)(2n+1) - (1^2 + 2^2 + 3^2 + \cdots + n^2) \\
&\quad = \frac{1}{2}n(n+1)(2n+1) - \frac{1}{6}n(n+1)(2n+1) \\
&\quad = \frac{1}{3}n(n+1)(2n+1).
\end{aligned}$$

Q1137 At a party, there are $2n$ people, of whom each one is acquainted with at least n others. Prove that it is possible to seat them at a circular table so that each is seated between two acquaintances.

ANS. (1) We first prove that it is possible to seat at least $n + 1$ round the table, each flanked by 2 acquaintances. Stand any person on a line, next to him place an acquaintance, then one of his acquaintances, and so on, adding at either end of the line until it is impossible to obtain a longer line in which each person knows both his neighbours. Call the person at one end of the line A. Then all of A's acquaintances are already in the line. The section of the line from A to the most distant of his friends therefore contains at least $n + 1$ people, and these may be seated round the table in the order in which they stand in the line.

(2) Suppose now that k is the largest number of people which can be seated round the table so that each knows both neighbours. Because of (1) $k \geq n + 1$. We are asked to

prove that $k = 2n$. Suppose that $k < 2n$. If B is one of the remaining $2n - k$ people, since $2n-k \leq n-1$, some of B's acquaintances must already be seated. We can stand B behind a seated acquaintance, P, then (if there is one standing) another acquaintance of B behind him and so on, to obtain a queue of people standing behind one of the seated group so that each person in the queue knows his neighbours and so that the queue cannot be lengthened further. Suppose there are ℓ people standing in the queue, where $\ell \leq n - k$. If the person at the tail of the queue is C, all of his acquaintances must already be either seated, or in front of him in the queue. A simple calculation shows that C must have at least $n - (\ell - 1) - 1 = (n - \ell)^*$ acquaintances other than P who are already seated. Let S be the set of ℓ people seated to the right of P. None of them (D say) could be acquaintances of C since otherwise we could replace the people between P and D by the people standing in the queue to obtain more than k people seated satisfactorily, contradicting the maximality of k. The same argument of course applied to the ℓ people T seated to the left of P. Hence all of C's seated acquaintances are included amongst the $k - 2\ k - 2\ell - 1$ people, U, more than ℓ seats away from P. Furthermore, no two adjacent people seated can both be acquainted with C, since we could simply seat C between them to again contradict the maximality of k. If at most every second person in U is acquainted with C, his seated acquaintances in the arc are in number $\leq \frac{(k-2\ell-1)+1}{2}$.

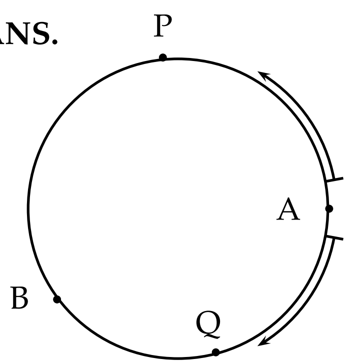
$\therefore$ Using *, $n - \ell \leq \frac{k}{2} - \ell$

$\therefore\ k \geq 2n$.

Hence we must have $k = 2n$ as required.

Q1138 Two bodies move in opposite directions around a circular track, one at constant speed v m/sec. The speed of the other increases at a constant rate a m/sec/sec. At time $t = 0$, the bodies are at the same point A and the second one is momentarily at rest. In how many seconds does their first meeting occur, if their second meeting is again at the point A?

ANS.



Let d be the distance (in metres) round the track, and suppose their first meeting, at B, occurred after t_1 secs. Then

$$\begin{aligned} d &= \text{length } APB + \text{ length } AQB \\ &= vt_1 + \tfrac{1}{2}at_1^2. \end{aligned} \tag{2}$$

Since their second meeting, after t_2 secs, occurred at A,

$$d = vt_2 \tag{3}$$

and

$$d = \tfrac{1}{2}at_2^2. \tag{4}$$

From (3) and (4) $d = \frac{1}{2}a\left(\frac{d}{v}\right)^2$, yielding $d = \frac{2v^2}{a}$.

Substitution in (2) gives $\frac{1}{2}at_1^2 + vt_1 - \frac{2v^2}{a} = 0$ from which $t_1 = \frac{v}{a}(-1+\sqrt{5})$, since the negative root must be rejected.

Hence the first meeting occurred after $\frac{v}{a}(\sqrt{5}-1)$ seconds.

Q1139 A co-ed school organizes a chess competition in which every participant plays one game against every one else. At the conclusion of the tournament, it is noted that each competitor gained exactly half of his or her points in games against boys. (As usual, a win earns 1 point, and a draw earns $\frac{1}{2}$ point for both players.) Prove that the total number of competitors is a perfect square.

ANS. It is clear that for any set of competitors, S, the total number of points earned against boys by all members of S is equal to the total number of points they have earned against girls(*).

Let b denote the number of boys competing, and g the number of girls. Then $b(b-1)/2$ games are played between boys, $g(g-1)/2$ between girls, and bg between a boy and a girl. Let w denote the number of points earned by boys against girls. Then the number of points earned by girls against boys must be $bg-w$. Applying the observation (*) first taking S to be the set of all boys, then the set of all girls, gives the two equations:

$$\begin{aligned} b(b-1)/2 &= w \\ bg - w &= g(g-1)/2. \end{aligned}$$

Eliminating w yields, after transposition of terms

$$\begin{aligned} b+g &= b^2 - 2bg + g^2 \\ &= (b-g)^2 \end{aligned}$$

Hence the total number of competitors, $b+g$, is a perfect square.

Q1140 Fewer than 135 players entered a tournament in which each match has three contestants and one winner. Before each round the names of all the survivors were drawn to provide as many matches as possible, with byes being given to any players left out. It turned out that the tournament was won outright by a player who contested only one match. What was the greatest possible number who took part?

ANS. The greatest number who took part is 109.

Since there was an outright winner, there were precisely three players left with one round to go, one of whom was the eventual winner, this being his only match. So in the earlier rounds he, at least, was given a bye, so at no earlier stage was the number of players divisible by 3. Indeed, if, after any round but the last, there were n players left, then before that round there must have been either $2+3(n-2) = 3n-4$ or $1+3(n-1) = 3n-2$ players.

The "tree" below shows the possible numbers of players who entered the tournament so that the above might occur.

Since there were fewer than 135 players, the greatest possible number who took part was 109.