

The Harmonic Series and Euler's Constant

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If you study series, one of the first divergent series you will meet is the harmonic series,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k}.$$

There are many ways to see that the harmonic series diverges. I will show you two, completely different ways. To begin with I will show you a proof by contradiction, i.e., I will assume that the series converges and then deduce a contradiction showing that the assumption that the series converges is false. Suppose the series converges, and its sum is S . Then

$$\begin{aligned} S &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8}\right) + \cdots \\ &> \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \cdots \\ &= \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \\ &= \frac{1}{2} + S, \end{aligned}$$

where I have replaced each fraction with an odd denominator greater than or equal to 3 with the smaller fraction with denominator increased by 1, and in the next step I have used the fact that

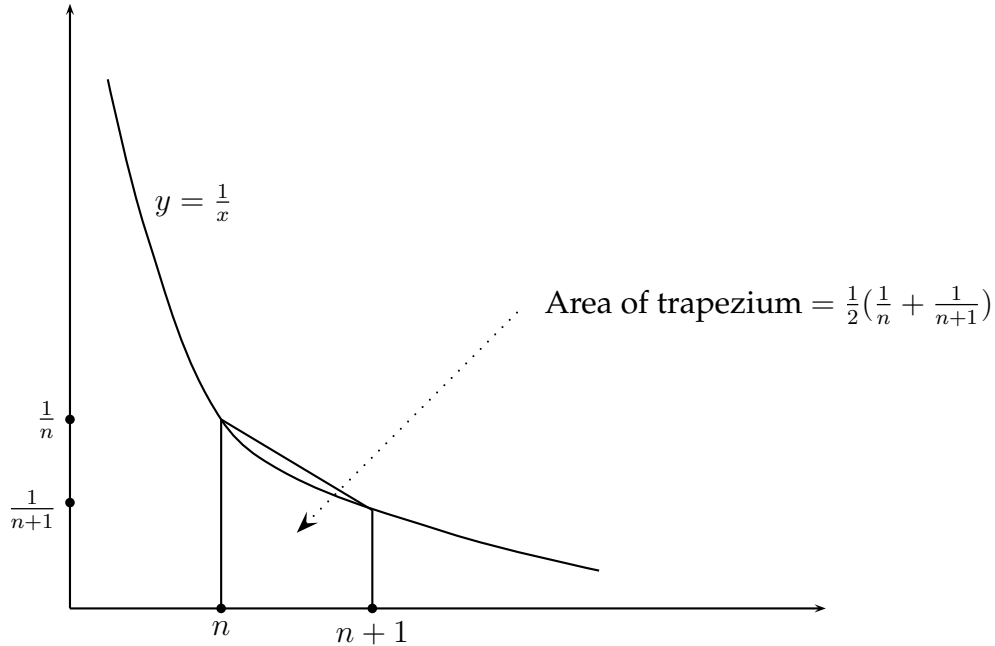
$$\frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$

Thus we obtain $S > \frac{1}{2} + S$, which you will agree is impossible. The conclusion is that the harmonic series does not add up to a number, S . That is, the series diverges.

My next demonstration that the harmonic series diverges uses integration. Thus it is rather more sophisticated than my first demonstration.

Draw the graph of $y = f(x) = 1/x$ for x from 1 to ∞ . Above each interval $[n, n+1]$, $n = 1, 2, 3, \dots$, draw the line that joins the point $(n, 1/n)$ to the point $(n+1, 1/(n+1))$. This forms a series of trapezia and the area of a general trapezium on the interval $[n, n+1]$ is simply $\frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+1} \right) + \frac{1}{n+1} = \frac{1}{2} \left(\frac{1}{n} + \frac{1}{n+1} \right)$.

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From the figure above it is clear that $f(x)$ lies below the tops of the trapezia so

$$\int_1^N f(x) dx < \text{sum of the areas of the trapezia} = S_N$$

if N is an integer greater than 1.

That is,

$$\log_e N < \frac{1}{2} \left(1 + \frac{1}{2} \right) + \frac{1}{2} \left(\frac{1}{2} + \frac{1}{3} \right) + \cdots + \frac{1}{2} \left(\frac{1}{N-1} + \frac{1}{N} \right),$$

or, if we add both $\frac{1}{2}$ and $\frac{1}{2N}$ to both sides,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} > \log_e N + \frac{1}{2} + \frac{1}{2N}.$$

Since $\log_e N \rightarrow \infty$ as $N \rightarrow \infty$, we see that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \rightarrow \infty \text{ as } N \rightarrow \infty,$$

that is, the harmonic series diverges.

We saw above that the difference between S_N and $\int_1^N f(x) dx$ is pretty small (the two graphs are close together), and indeed that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} - \log_e N$$

is a little more than $\frac{1}{2}$. What I would like to do now is study that quantity, so let me define

$$\delta(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log_e n.$$

We saw that

$$\delta(n) > \frac{1}{2}.$$

What I will show now is that $\delta(n)$ is a decreasing function of n , and so $\delta(n)$ is never more than 1, its value when $n = 1$.

Let us examine the quantity

$$\begin{aligned} \delta(n) - \delta(n+1) &= \log_e(n+1) - \log_e n - \frac{1}{n+1} \\ &= \int_n^{n+1} \left(\frac{1}{x} - \frac{1}{n+1} \right) dx \\ &> 0 \end{aligned}$$

since the integrand is greater than 0 for $n \leq x < n+1$ (and equals 0 at $x = n+1$). So $\delta(n)$ is a decreasing quantity and is bounded below by $\frac{1}{2}$. It follows that $\delta(n)$ approaches a limit as $n \rightarrow \infty$, and that limit is greater than or equal to $\frac{1}{2}$. The limit as $n \rightarrow \infty$ of $\delta(n)$ is called "Euler's constant", and is denoted by γ'

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log_e n \right) = \gamma \geq \frac{1}{2}.$$

Euler introduced the constant γ in 1735 and calculated it to sixteen digits in 1781. There are many remarkable mathematical relations involving γ . One such relation discovered recently is $e^\gamma = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1.3}\right)^{1/3} \left(\frac{2^3 \cdot 2^4}{1.3^3}\right)^{1/4} \cdots$ Sondow (2003).

The remainder of this article is devoted to getting a really good approximation to γ , and a really good estimate of the difference between $\delta(n)$ and γ . First I want to re-examine the quantity $\delta(n) - \delta(n+1)$. We have

$$\begin{aligned} \delta(n) - \delta(n+1) &= \log_e(n+1) - \log_e(n) - \frac{1}{n+1} \\ &= \int_n^{n+1} f(x) dx - \frac{1}{n+1} < \int_n^{n+1} g(x) dx - \frac{1}{n+1} \\ &= \frac{1}{2} \left(\frac{1}{n} + \frac{1}{n+1} \right) - \frac{1}{n+1} \\ &= \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+1} \right). \end{aligned}$$

If we write this down for $n, n + 1, n + 2, \dots, N - 1$, we get

$$\begin{aligned}\delta(n) - \delta(n + 1) &< \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n + 1} \right), \\ \delta(n + 1) - \delta(n + 2) &< \frac{1}{2} \left(\frac{1}{n + 1} - \frac{1}{n + 2} \right), \\ \delta(n + 2) - \delta(n + 3) &< \frac{1}{2} \left(\frac{1}{n + 2} - \frac{1}{n + 3} \right), \\ &\vdots \\ \delta(N - 1) - \delta(N) &< \frac{1}{2} \left(\frac{1}{N - 1} - \frac{1}{N} \right).\end{aligned}$$

If we add all these up, we get

$$\delta(n) - \delta(N) < \frac{1}{2} \left(\frac{1}{n} - \frac{1}{N} \right).$$

If we now hold n fixed and let $N \rightarrow \infty$, and remember that $\lim_{N \rightarrow \infty} \delta(N) = \gamma$, we find that

$$\delta(n) - \gamma \leq \frac{1}{2n}.$$

That is,

$$\gamma \geq \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log_e n \right) - \frac{1}{2n}.$$

If we put $n = 1$, we get $\gamma \geq \frac{1}{2}$, which is no more than we knew before, but if we put $n = 100$, and use the computer to calculate the right hand side, we get

$$\gamma \geq 0.577207332,$$

which, as we shall see, is not a bad start! To do more, it seems to me that we have to know more. The first thing we have to know is the series for $\log_e(1 + x)$,

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots,$$

valid for $0 \leq x \leq 1$. (It's actually also valid for $-1 < x < 0$, and in the complex domain for $|x| < 1$, but we don't need that.) This is a remarkable formula, which, as you will see, is not hard to prove. We start with the sum to n terms of the geometric series

$$1 - t + t^2 - \dots + (-1)^{n-1} t^{n-1} = \frac{1 - (-t)^n}{1 + t}.$$

We can rearrange this as follows.

$$\frac{1}{1 + t} = 1 - t + t^2 - \dots + (-1)^{n-1} t^{n-1} + (-1)^n \frac{t^n}{1 + t}.$$

If we now integrate this from 0 to x , we get

$$\begin{aligned}\log_e(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + (-1)^n \int_0^x \frac{t^n}{1+t} dt \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + (-1)^n \epsilon_n\end{aligned}$$

where

$$0 \leq \epsilon_n = \int_0^x \frac{t^n}{1+t} dt \leq \int_0^x t^n dt = \frac{x^n}{n}.$$

If we fix x in the interval $0 \leq x \leq 1$ and let $n \rightarrow \infty$, we see that $\epsilon_n \rightarrow 0$, and so

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

as I claimed above. We found above that

$$\begin{aligned}\delta(n) - \delta(n+1) &= \log_e(n+1) - \log_e n - \frac{1}{n+1} \\ &= \log_e \left(1 + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + - \cdots\right) \\ &\quad - \left(\frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^3} - \frac{1}{n^4} + - \cdots\right) \\ &= \frac{1}{2n^2} - \frac{2}{3n^3} + \frac{3}{4n^4} - + \cdots,\end{aligned}$$

this being valid for $n > 1$. **Indeed, from now on, I shall assume that $n > 1$.** On the other hand, we have that

$$\begin{aligned}\frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+1}\right) &= \frac{1}{2} \left(\frac{1}{n} - \left(\frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^3} - \frac{1}{n^4} + - \cdots\right)\right) \\ &= \frac{1}{2} \left(\frac{1}{n^2} - \frac{1}{n^3} + \frac{1}{n^4} - + \cdots\right) \\ &= \frac{1}{2n^2} - \frac{1}{2n^3} + \frac{1}{2n^4} - + \cdots.\end{aligned}$$

The difference between the two series is

$$\begin{aligned}&\left(\frac{1}{2n^2} - \frac{1}{2n^3} + \frac{1}{2n^4} - \frac{1}{2n^5} - + \cdots\right) \\ &\quad - \left(\frac{1}{2n^2} - \frac{2}{3n^3} + \frac{3}{4n^4} - \frac{4}{5n^5} + - \cdots\right) \\ &= \frac{1}{6n^3} - \frac{2}{8n^4} + \frac{3}{10n^5} - \frac{4}{12n^6} + - \cdots.\end{aligned}$$

That is,

$$\delta(n) - \delta(n+1) = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+1} \right) - \left(\frac{1}{6n^3} - \frac{2}{8n^4} + \frac{3}{10n^5} - \frac{4}{12n^6} + - \dots \right).$$

Now, somewhat out of the blue, consider the series for

$$\begin{aligned} \frac{1}{12} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) &= \frac{1}{12} \left(\frac{1}{n^2} - \left(\frac{1}{n^2} - \frac{2}{n^3} + \frac{3}{n^4} - \frac{4}{n^5} + - \dots \right) \right) \\ &= \frac{2}{12n^3} - \frac{3}{12n^4} + \frac{4}{12n^5} - \frac{5}{12n^6} + - \dots . \end{aligned}$$

From the last two series, it is clear that

$$\begin{aligned} \delta(n) - \delta(n+1) &= \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+1} \right) - \frac{1}{12} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \\ &\quad + \left(\frac{4}{12} - \frac{3}{10} \right) \frac{1}{n^5} - \left(\frac{5}{12} - \frac{4}{12} \right) \frac{1}{n^6} \\ &\quad + \left(\frac{6}{12} - \frac{5}{14} \right) \frac{1}{n^7} - + \dots . \end{aligned}$$

A careful analysis of this enables us to say that

$$\delta(n) - \delta(n+1) > \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+1} \right) - \frac{1}{12} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right).$$

If we write this for $n, n+1, \dots, N-1$ and add, we get

$$\delta(n) - \delta(N) > \frac{1}{2} \left(\frac{1}{n} - \frac{1}{N} \right) - \frac{1}{12} \left(\frac{1}{n^2} - \frac{1}{N^2} \right).$$

If we fix n and let $N \rightarrow \infty$, we get

$$\delta(n) - \gamma \geq \frac{1}{2n} - \frac{1}{12n^2},$$

or,

$$\gamma \leq \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log_e n \right) - \frac{1}{2n} + \frac{1}{12n^2}.$$

Putting $n = 100$ gives

$$\gamma \leq 0.577215665$$

We can continue in the same way, and find that

$$\delta(n) - \gamma \geq \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6},$$

and

$$\delta(n) - \gamma \leq \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \frac{1}{240n^8}$$

and so on.

If we put $n = 100$ in the last two of these we find

$$0.5772156649015328606 \leq \gamma \leq 0.5772156649015328611.$$

There are more things that can be said.

(a) Although we can calculate γ to any large finite number of decimal places, no-one knows whether γ is irrational or rational. (b) The pattern above can be continued, and involves numbers called the Bernoulli numbers. I had never seen this before, but I guess someone found it before me.

$$(c) \quad S_n - \int_1^n f(x) dx = \delta(n) - \frac{1}{2} - \frac{1}{2n}$$

is an increasing quantity, approaches $\gamma - \frac{1}{2}$ as $n \rightarrow \infty$, and from the work we have done above can be put between bounds involving only even powers of n .