On Iterated Exponentiation – The Hyperexponentials

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Introduction

Sometime in your senior mathematics course you will have come across arithmetic and geometric sequences. As examples of these two sequences we have,

$$\{nx\} = x, 2x, 3x, \ldots$$

for an arithmetic sequence, and

$$\{x^n\} = x, x^2, x^3, \ldots$$

for a geometric sequence. Here, in both cases, n is a positive integer. Notice that each consecutive term in the above arithmetic sequence is obtained by adding x to its proceeding term. This process of adding x to each proceeding term in order to form the next term in the sequence is called repeated or *iterated addition*. Similarly, in the above geometric sequence each consecutive term is obtained by multiplying x to its proceeding term in a process called *iterated multiplication*. Now imagine the sequence which would be formed by exponentiating x to its proceeding term in a process which is referred to as *iterated exponentiation*. Here a *hyperexponential* sequence whose terms consists of hyperexponents (or *hyperpowers*) in x would result. It is the terms in this sequence, and some of their associated properties, which this article will focus on.

The hyperexponentials

In order to establish the relevant notation, we define the hyperexponential (or *power tower*) of order n as

$$^{n}x \equiv \underbrace{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}_{n}.$$

Thus ^{n}x consists of n towers of x. The hyperexponential sequence, which is formed by the process of iterated exponentiation, can therefore be written as

$$\begin{aligned}\{^{n}x\} &= {}^{1}x, {}^{2}x, {}^{3}x, \ldots \\ &= x, x^{x}, x^{x^{x}}, \ldots\end{aligned}$$

Here it is usual for the index n to be referred to as the hyperexponent.

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In order to better understand where the hyperexponential sequence has come from, let's step carefully through the processes which we have taken in order to arrive at each sequence up to the hyperexponential sequence. We see in the arithmetic sequence that multiplication is just repeated addition (e.g. $3x = x+x+x$). In the geometric sequence, exponentiation is just repeated multiplication (e.g. $x^3 = x \times x \times x$). So in the hyperexponential sequence *tetration* is the operation which results from repeated exponentiation (e.g. $^3x = x^{x^x}$). The word "tetra" stems from the Greek "tettares" meaning four. Thus the word *tetration* is used for the process of repeated exponentiation since it can be seen that tetration is the fourth operation in the sequence: addition, multiplication, exponentiation, and ... tetration. Extending this idea beyond tetration, it is not hard to see that we would have "pentation" and so on.

Returning to the hyperexponentials, we see that we are now able to read ^nx as 'x tetrated to the n'. It is also important to use the proper convention and read the powers downwards. So, for example, four tetrated to the three is $^34 = 4^{4^4} = 4^{256}$ and not 256^4.

The following table gives some values for the first few positive integer hyperexponentials:

	^nx	$n \longrightarrow$			
		1	2	3	4
x	1	1	1	1	1
$\downarrow$	2	2	4	16	65 536
	3	3	27	7.63×10^{12}	
	4	4	256	1.34×10^{154}	

In obtaining these values a pocket calculator can be used to calculate the first half a dozen or so. You will notice, however, that the capacity of the pocket calculator for directly calculating such hyperexponentials is quickly exceeded after only the first few have been calculated. The next few can then be calculated using a pocket calculator and logarithms to base 10. For example, consider

$$^34 = 4^{4^4} = 4^{256}.$$

Taking base 10 logarithms of both sides of the above equation gives

$$\begin{aligned} \log_{10}\left(^34\right) &= \log_{10}\left(4^{256}\right) \\ &= 256 \log_{10}(4) \\ &= 256 \times 0.60205999\ldots \\ &= 154.1273578\ldots \\ &= 154 + 0.1273578\ldots \end{aligned}$$

Exponentiating both sides to the power of 10 gives

$$\begin{aligned} ^34 &= 10^{154+0.1273578\ldots} \\ &= 10^{0.1273578\ldots} \times 10^{154} \\ &= (1.34078\ldots) \times 10^{154} \\ \therefore {}^34 &\simeq 1.34 \times 10^{154} \end{aligned}$$

It is however not long before even this method reaches its limits. As an exercise you may like to try to calculate $^5 2$.

Do not be too surprised if you have not heard of the hyperexponentials before. In general, little is heard about them since as we saw above, tetrating even the smallest of numbers produces inordinately large numbers! However, for values between zero and one, not only are the values of the hyperexponentials to any order small, but interesting behaviour is seen. This will be taken up and explored further in the next section.

Exploring the notation for tetration further, we might ask ourselves the question as to what exactly does $^m(^n x)$ equal? This is nothing more than a tetration of a tetration. We have

$$^m(^n x) = \overbrace{^m\left(x^{x^{\cdot^{\cdot^{\cdot^x}}}}\right)}^{n}$$

$$= \underbrace{\left(x^{x^{\cdot^{\cdot^{\cdot^x}}}}\right)^{\overbrace{\left(x^{x^{\cdot^{\cdot^{\cdot^x}}}}\right)}^{n}^{\cdot^{\cdot^{\cdot}}\overbrace{\left(x^{x^{\cdot^{\cdot^{\cdot^x}}}}\right)}^{n}}}_{\underbrace{n}}}_{m}$$

that is, m towers of $\underbrace{\left(x^{x^{\cdot^{\cdot^{\cdot^x}}}}\right)}_{n}$. It can also be seen that in general

$$^m(^n x) \neq {}^n(^m x),$$

unless $m = n$ or one of m, n or x is equal to unity. As an example consider $^3(^2 2)$. Here $^3(^2 2)$ consists of 3 towers of 2^2, namely

$$^3(^2 2) = {}^3(2^2) = \left(2^2\right)^{\left(2^2\right)^{\left(2^2\right)}} = 4^{4^4} = 4^{256} \simeq 1.34 \times 10^{154}.$$

Compare this to $^2(^3 2)$, which consists of 2 towers of 2^{2^2}. Thus

$$^2(^3 2) = {}^2\left(2^{2^2}\right) = \left(2^{2^2}\right)^{\left(2^{2^2}\right)} = 16^{16} \simeq 1.84 \times 10^{19}.$$

As a final point on notation it should be clear that $^m(^n x)$ is not the same operation as $^{mn} x$. Here the former is a tetration of a tetration and consists of m towers of $^n x$ while the latter is a single tetration and consists of $m \times n$ towers of x.

The hyperexponential functions

Each term in the hyperexponential sequence represents a real function in x if $x \geq 0$. This class of functions are usually called either the (finite) hyperexponential or hyper-power functions. This class of hyperexponential functions can be denoted by

$$h(x, n) = {}^{n}x.$$

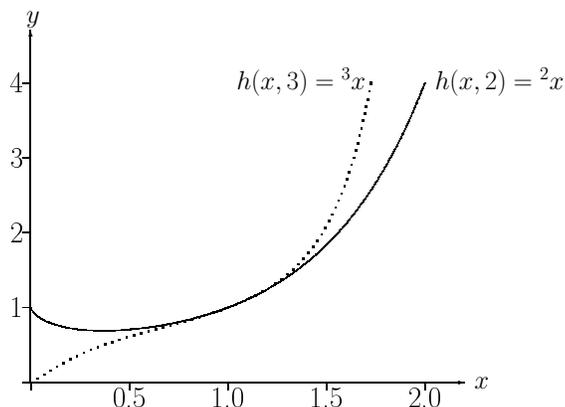Here it is important to remember that n is a finite, non-zero, positive integer.

Figure 1: Graph of the second and third hyperexponential functions

In figure Figure 1, $h(x, 2) = {}^{2}x$ (the second) and $h(x, 3) = {}^{3}x$ (the third) hyperexponential functions are plotted. Notice that both curves pass through the point $(1, 1)$. In fact, for all the hyperexponential functions to any order it is easy to see that we must have $h(1, n) = 1$. Also notice the behaviour of each of the curves between zero and one. There seems to be a natural divide between the two curves depending on whether n is even or odd. Finally, it should be obvious that in the limit of very large x, the hyperexponential curves for all orders in the hyperexponent n rapidly tend to infinity.

In figure 2, ${}^{2}x$ through to ${}^{9}x$ are plotted in the interval $[0, 1]$.
Figure 2 shows quite clearly that the hyperexponential functions divide into two subsequences depending on whether the hyperexponent n is even or odd. Writing then

$$\{{}^{2k}x\} = {}^{2}x, {}^{4}x, {}^{6}x, \ldots$$

and

$$\{{}^{2k-1}x\} = {}^{1}x, {}^{3}x, {}^{5}x, \ldots$$

where k is a non-zero, positive integer. Figure 2 seems to suggest that

$$\lim_{x \to 0^{+}} {}^{2k}x = 1, \quad \text{and} \quad \lim_{x \to 0^{+}} {}^{2k-1}x = 0.$$

Both of these one-sided limits are true for all orders in their hyperexponents. They can be formally proved using a rule for limits which is known as L'Hôpital's Rule and then
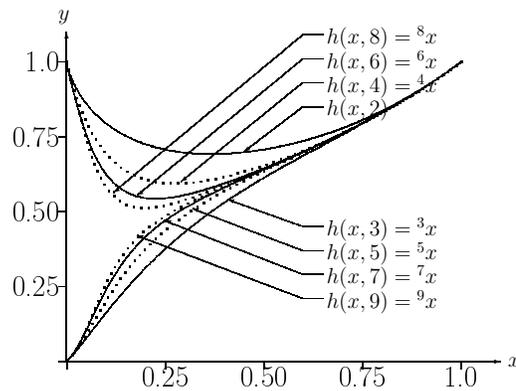
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Figure 2: Graph of the second and third hyperexponential functions

induction [1]. You may like to check the validity of the first few limits using a pocket calculator and a very small, positive number which is close to zero.

A little calculus

Let's now turn to the question of differentiating expressions containing the hyperexponential functions. Here we wish to find general expressions for the first and second derivatives of $h(x, n) = {}^{n}x$. We will then use these derivatives to analyse the behaviour of the hyperexponential curves in the interval $[0, 1]$.

The first derivative of ${}^{n}x$ can readily be found by taking logarithms of both sides of the equation $y = {}^{n}x$ and then differentiating implicitly. So, taking logarithms of both sides of the equation $y = {}^{n}x$ gives

$$
\begin{aligned}
\ln y &= \ln {}^{n}x \\
&= \ln \underbrace{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}_{n} \\
&= \underbrace{x^{\cdot^{\cdot^{\cdot^{x}}}}}_{n-1} \ln x \\
&= {}^{n-1}x \ln x,
\end{aligned}
$$

which upon differentiating implicitly with respect to x gives

$$
\frac{d}{dx}\left({}^{n}x\right) = {}^{n}x \left[\ln x \frac{d}{dx}\left({}^{n-1}x\right) + \frac{{}^{n-1}x}{x}\right]. \tag{1}
$$

This is a recurrence formula for the first derivatives of the hyperexponentials to all orders of the hyperexponent $n > 1$. For $n = 1$ we recognise that

$$
\frac{d}{dx}\left({}^{1}x\right) = \frac{d}{dx}(x) = 1.
$$

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The first few derivatives are:

$$\frac{d}{dx}\left(^{2}x\right) = {}^{2}x(\ln x + 1) = x^{x}(\ln x + 1), \quad \text{and}$$

$$\frac{d}{dx}\left(^{3}x\right) = {}^{3}x.{}^{2}x\left[(\ln x + 1)\ln x + \frac{1}{x}\right] = x^{x^{x}+x-1}\left[x\ln x(\ln x + 1) + 1\right].$$

The second derivative of the hyperexponential functions is found by differentiating equation (1) with respect to x. It is a fairly straight forward, but tedious, exercise to show that the second derivative will be given by the recurrence relation

$$\frac{d^{2}}{dx^{2}}\left(^{n}x\right) = {}^{n}x\left[\ln x \frac{d}{dx}\left(^{n-1}x\right) + \frac{^{n-1}x}{x}\right]^{2}$$

$$+{}^{n}x\left[\ln x \frac{d^{2}}{dx^{2}}\left(^{n-1}x\right) + \frac{2}{x}\frac{d}{dx}\left(^{n}x\right) - \frac{^{n-1}x}{x^{2}}\right], \tag{2}$$

for $n > 1$ while $\frac{d^{2}}{dx^{2}}\left(^{1}x\right) = 0$ for $n = 1$. The first few second derivatives are:

$$\frac{d^{2}}{dx^{2}}\left(^{2}x\right) = {}^{2}x\left[(\ln x + 1)^{2} + \frac{1}{x}\right], \quad \text{and}$$

$$\frac{d^{2}}{dx^{2}}\left(^{3}x\right) = {}^{3}x.{}^{2}x\left[\ln x(\ln x + 1) + \frac{1}{x}\right]^{2}$$

$$+{}^{3}x.{}^{2}x\left[\ln x\left((\ln x + 1)^{2} + \frac{1}{x}\right) + \frac{2(\ln x + 1)}{x} - \frac{1}{x^{2}}\right].$$

Knowledge of the first and second derivatives of the hyperexponential functions can now be used in an analysis for finding stationary points and possible points of inflection in the curves of ^{n}x. Unfortunately, such an analysis using analytic methods is all but impossible except for the hyperexponential function $h(x, 2) = {}^{2}x$. Instead, either a graphics calculator or a computer algebra system, such as MAPLE, needs to be used.

No minima are ever found for any of the odd hyperexponential functions. On the other hand, each even hyperexponential function has a single mimima and it is interesting to note that these minima approach the bifurcation point $x^{\star} = 1/e^{e}$ as n increases. To see this, a few of the minimum turning points obtained from solving $\frac{d}{dx}\left(^{n}x\right) = 0$ for $n = 2, 4, 10, 20$ and 40 are given below.

For points of inflection, a far more complicated picture emerges. It turns out that the hyperexponential functions can have between none and a number of such points depending on the value of the hyperexponent. Such details, however, are only borne out through detailed numerical analysis. You may like to try some of this initial numerical analysis yourself if you have access to a computer algebra system. For a more in-depth treatment of some of the critical points of the hyperexponential functions, the interested reader is directed to an article by MacDonnell [3].

n	Minimum turning point	Value of ^{n}x at the turning point
2	$e^{-1} = 0.36787\ldots$	$e^{-e^{-1}} = 0.69220\ldots$
4	$0.27468\ldots$	$0.59323\ldots$
10	$0.17507\ldots$	$0.49073\ldots$
20	$0.12786\ldots$	$0.44045\ldots$
40	$0.10009\ldots$	$0.40916\ldots$
$\vdots$	$\vdots$	$\vdots$
∞	$e^{-e} = 0.06598\ldots$	$e^{-1} = 0.36787\ldots$

Table 1: Minimum turning points for a few of the even hyperexponential functions.

The odd hyperexponential functions

From figure 2 it appears as though those hyperexponential curves with odd hyper-exponents are increasing functions on $x > 0$. Recall that a function $f(x)$ is said to be *increasing* if $f'(x) > 0$ for all values of x in its domain where this inequality holds. Geometrically, the curve of an increasing function slopes upwards so that a tangent drawn to the curve has a positive gradient.

We now wish to show that all odd hyperexponential functions are increasing functions for $x > 0$. To do this we will use equation (1) and induction. This proves to be an interesting exercise in induction on the hyperexponentials. Before commencing with the proof however, we need to establish, at least in a non-rigorous manner, that

$$^{n}x > 1/e, \tag{3}$$

for $x > 0$ and even positive integers n.

In the previous section we saw that each even hyperexponential function had a single minimum and this minimum point approached the bifurcation point of $1/e^e$ with increasing n such that the value of ^{n}x tended to $1/e$ from above (see table 1). Consequentially, we can conclude that all hyperexponential functions of (finite) even order will have a value greater than $1/e$ for $x > 0$. On taking the logarithm of both sides of the inequality given by equation (3), and rearranging, we have the equivalent inequality of

$$1 + \ln(^{n}x) > 0.$$

For convenience we let X_n denote the first derivative of ^{n}x. So from equation (1) we have

$$X_n \equiv \frac{d}{dx}\left(^{n}x\right) = {}^{n}x\left[\ln x \frac{d}{dx}\left(^{n-1}x\right) + \frac{^{n-1}x}{x}\right].$$

For the hyperexponentials it should also be obvious that ^{n}x can be defined recursively as

$$^{n}x = x^{^{n-1}x} \quad \text{for} \quad n > 1,$$

so that

$$\ln \left({}^{n}x\right) = {}^{n-1}x \ln x.$$

We are now in a position to proceed with the proof, by induction, that all hyperexponential curves with odd hyperexponents are increasing functions on $x > 0$.

Clearly, when $n = 1$, $X_1 = \dfrac{d}{dx}({}^{1}x) = \dfrac{d}{dx}(x) = 1 > 0$ for $x > 0$. So the statement is true for $n = 1$. Now suppose that $k \geq 2$ is an even positive integer for which the statement is true, namely ${}^{k-1}x$ is increasing. That is, suppose

$$X_{k-1} = {}^{k-1}x \left[X_{k-2} \ln x + \frac{{}^{k-2}x}{x} \right] > 0.$$

Now proving this statement true for $n = k+2$. That is, we need to prove that $X_{k+1} > 0$. So

$$
\begin{aligned}
X_{k+1} &= {}^{k+1}x \left[X_k \ln x + \frac{{}^{k}x}{x} \right] \\
&= X_k \, {}^{k+1}x \ln x + \frac{{}^{k+1}x \, {}^{k}x}{x} \\
&= {}^{k}x \left[X_{k-1} \ln x + \frac{{}^{k-1}x}{x} \right] {}^{k+1}x \ln x + \frac{{}^{k+1}x \, {}^{k}x}{x} \\
&= {}^{k}x \, {}^{k+1}x \left[X_{k-1}(\ln x)^2 + \frac{{}^{k-1}x \ln x}{x} + \frac{1}{x} \right] \\
&= {}^{k}x \, {}^{k+1}x \left[X_{k-1}(\ln x)^2 + \frac{1 + \ln\left({}^{k}x\right)}{x} \right],
\end{aligned}
$$

since ${}^{k-1}x \ln x = \ln\left({}^{k}x\right)$. Now for $x > 0$, clearly ${}^{k}x, {}^{k+1}x > 0$ and $X_{k-1} > 0$ by the induction hypothesis. Also, since $1 + \ln\left({}^{k}x\right) > 0$ when k is an even integer, then it follows that $X_{k+1} > 0$. Thus ${}^{k+1}x$ is increasing. So by induction, for each odd positive integer n on $x > 0$, ${}^{n}x$ will be increasing.

References

[1] J. Marshall Ash, "The limit of $x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}$ as x tends to zero," *Math. Mag.* **69** (3), 207–209 (1996).

[2] R. A. Knoebel, "Exponential reiterated," *Amer. Math. Monthly* **88** (4), 235–252 (1981).

[3] J. F. MacDonnell, "Some critical points on the hyperpower function ${}^{n}x = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$," *Int. J. Math. Educ. Sci. Technol.* **20** (2) 297–305 (1989).