Parabola Volume 40, Issue 2 (2004)

UNSW School Mathematics Competition 2004

Problems and Solutions

Junior Division

1. In how many ways can a cube be coloured with the three colours red, white and green? (Each face is to be one colour, of course, but you don't have to use all three colours. Two cubes are considered to be coloured the same way if you can rotate them and position them in such a way that the colours are in matching positions on the two cubes.)

   Solution:

   There are 57 different ways of colouring a cube.

   Using one colour, there are three.

   Using two colours, 5 faces one colour, 1 the other, there are six choices for the colours, and just one arrangement, so 6 cubes.

   With 4 faces one colour, 2 the other, there are six choices of colours, and two arrangements (the two faces of the one colour can be opposite or adjacent), so 12 cubes.

   With 3 faces one colour, 3 the other, there are three choices of colours and two arrangements (the faces of each colour are adjacent or in a strip), so 6 cubes.

   Using three colours, 4 faces of one colour and 1 each of the other two, there are three choices of colours and two arrangements (the single colours are opposite or adjacent), so 6 cubes.

   With 3 faces of one colour, 2 of a second and 1 of the third, there are six choices of colours, and for each choice there are three arrangements (the two faces of the second colour are opposite each other, or, if they are adjacent, the single face of the third colour can be adjacent to one or both faces of the second colour), so 18 cubes.

   With 2 faces of each colour, there are six arrangements. All three pairs of the same colour can be opposite, or just one pair can be opposing (three choices of colour) or none of the three pairs can be opposing, in which case there are **two** distinguishable arrangements, mirror images of one another! So 6 cubes. So the number of cubes is 3+6+12+6+6+18+6=57.

2. The numbers 1, 2, 3, 4 and 5 are separated into two sets. Show that, necessarily, one of the sets contains the difference between two of the numbers in that set. (The difference between two numbers is the bigger minus the smaller.)

Solution:

Suppose, to the contrary, that the numbers 1, 2, 3, 4, 5 can be separated into two sets in such a way that neither set contains the difference between two of its members. Then 1 and 2 belong to different sets, as do 2 and 4. So one set contains 1 and 4, the other 2. 5 cannot be in the same set as 1 and 4, so 5 is in with 2. 3 has nowhere to go! This is a contradiction, and proves the statement.

3. Find all solutions to

$$x^2 + y + z = 3, \;\; x + y^2 + z = 3, \;\; x + y + z^2 = 3.$$

Solution:

By subtracting pairs of the equations, we find $x(x-1) = y(y-1) = z(z-1)$. Since a quadratic can have the one value at most two points, we have that (at least) two of $x, \; y$ and z are equal. If all three are equal, we find they are all 1 or -3. If precisely two are equal, say $x = y$ then $z = 1 - x$ and we find $x = y = \pm\sqrt{2}, \;\; z = 1 \mp \sqrt{2}$. There are similar solutions with $x = z$, $y = z$.

4. Find all four–digit perfect squares with the property that the first (that is, left–most) digit is the same as the third, and the second is 4 more than the fourth.

Solution:

Let the first digit be a, the last d, and let the number be n^2. Then $n^2 = 1010a + 101d + 400$, $n^2 \equiv 20^2 \pmod{101}$, $n \equiv \pm 20 \pmod{101}$ (since 101 is prime). So $n = 20$ or $n = 81$. But we disallow the first, since we want $a \neq 0$. So $n = 81$, and the four–digit number is 6561.

5. A triangle is either isosceles (which for the purpose of this question we shall take to include equilateral) or scalene (all sides different).

   (a) How many triangles are there with integer–length sides and perimeter 24 ?

   (b) How many triangles are there with integer–length sides and perimeter 36 ?

   (c) How many isosceles triangles are there with integer–length sides and perimeter $12n$ (where n is an integer) ?

   (d) How many triangles are there altogether with integer–length sides and perimeter $12n$ (where n is an integer) ?

Solution:

   (a) There are 12 triangles with integer sides and perimeter 24.

With largest side 8, the other sides are 8, 8. With largest side 9, the other sides are 9, 6 or 8, 7. With largest side 10, the other sides are 10, 4 or 9, 5 or 8, 6 or 7, 7. With largest side 11, the other sides are 11, 2 or 10, 3 or 9, 4 or 8, 5 or 7, 6. The largest side cannot be more than 11. So there are 1+2+4+5=12 triangles.

(b) In the same way, we find that there are 1+2+4+5+7+8=27 triangles with integer sides and perimeter 36.

(c) There are $3n-1$ isosceles triangles with integer sides and perimeter $12n$ (including 1 equilateral triangle).

The "odd" side is even, and can be anything from 2 to $6n-2$. For each such, there is 1 isosceles triangle.

(d) There are $3n^2$ triangles with integer sides and perimeter $12n$.

The longest side is anything from $4n$ to $6n-1$. If the longest side is $4n+2k$ $(0 \le k \le n-1)$, the second–longest side is anything from $4n+2k$ down to $4n-k$, and there are $3k+1$ such triangles. So the number of triangles with longest side even is

$$\sum_{k=0}^{n-1} 3k+1 = \frac{1+(3n-2)}{2}n = (3n^2-n)/2.$$

If the longest side is $4n+1+2k$ $(0 \le k \le n-1)$, the second–longest side is anything from $4n+1+2k$ down to $4n-k$, and there are $3k+2$ such triangles. So the number of triangles with longest side odd is

$$\sum_{k=0}^{n-1} 3k+2 = \frac{2+(3n-1)}{2}n = (3n^2+n)/2.$$

6. The numbers a and b satisfy

$$a^3-3ab^2=52, \quad b^3-3a^2b=47.$$

Find a^2+b^2.

Solution:

$a^2+b^2=17$.

If you square both equations and add the two resulting equations, you get $(a^2+b^2)^3 = 52^2+47^2 = 2704+2209 = 4913 = 17^3$.

Senior Division

1. Solve for x, y and z,

$$x+\frac{1}{y}=1, \quad y+\frac{1}{z}=2, \quad z+\frac{1}{x}=5.$$

Solution:

We have

$$x + \cfrac{1}{2 - \cfrac{1}{5 - \cfrac{1}{x}}} = 1,$$

or,

$$(3x-1)^2 = 1.$$

It follows that $x = \frac{1}{3}$, $y = \frac{3}{2}$, $z = 2$.

2. (a) Two sequences are defined by

$$p_1 = 1, \;\; q_1 = 1 \;\text{ and for } n \geq 1, \;\; p_{n+1} = p_n + 2q_n, \;\; q_{n+1} = p_n + q_n.$$

Show that for all n, $p_n^2 - 2q_n^2 = (-1)^n$.

(b) Mike lives in a long street and his house has a 3–digit number. The odd–numbered houses are on one side of the street, the even–numbered on the other. Mike notices that the sum of the house–numbers on his side of the street up to and including his house is equal to the sum of the house–numbers on his side of the street from the other end of the street down to and including his house. What is Mike's house–number?

Solution:

(a) This can be proved by induction. The first few p_n and q_n are

p_n	1	3	7	17	41	99	239	577	1393	$\cdots$
q_n	1	2	5	12	29	70	169	408	985	$\cdots$

(b) Suppose Mike lives in an odd–numbered house, $2n-1$, and the last odd–numbered house in the street is $2m-1$. Then

$$n^2 = m^2 - (n-1)^2,$$

or

$$(2n-1)^2 - 2m^2 = -1.$$

Suitable values are given by the above table. Thus, $2n-1 = 239, \; m = 169$.

Similarly, if we suppose Mike's house is $2n$, and the last even–numbered house is $2m$, then

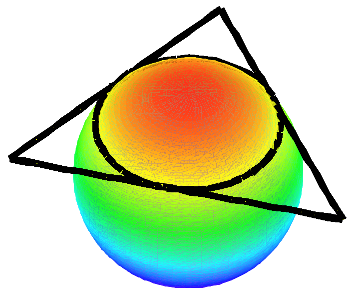
$$n(n+1) = m(m+1) - (n-1)n,$$

or

$$(2m+1)^2 - 2(2n)^2 = 1.$$

Suitable values are $2n = 408, \; 2m+1 = 577$.

Thus Mike's house–number is 239 or 408.



3. A triangle with sides 13, 14 and 15 sits around the top half of a sphere of radius 5 (that is, it touches the sphere at three points, as shown in the diagram above).

   How far is the plane of the triangle from the centre of the sphere?

   Solution:

   Let the sides of a triangle be a, b and c, and let $s = \dfrac{a+b+c}{2}$. Then the area of the triangle is

   $$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

   If the radius of the cirle is r, the area is

   $$A = rs.$$

   Thus

   $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

   Here, $a = 13$, $b = 14$, $c = 15$, so $s = 21$ and $r = 4$.

   It follows from Pythagoras' theorem that the plane of the circle is 3 units from the centre of the sphere.

4. A triangle is either isosceles (which for the purpose of this question we shall take to include equilateral) or scalene (all sides different).

   (a) How many triangles are there with integer–length sides and perimeter 24 ?

   (b) How many triangles are there with integer–length sides and perimeter 36 ?

   (c) How many isosceles triangles are there with integer–length sides and perimeter $12n$ (where n is an integer) ?

(d) How many triangles are there altogether with integer–length sides and perimeter $12n$ (where n is an integer) ?

Solution:

See the solution of Junior Question 5.

5. The numbers a and b satisfy

$$a^3 - 3ab^2 = 52, \;\; b^3 - 3a^2b = 47.$$

(a) Find $a^2 + b^2$

(b) Hence or otherwise find a and b.

(c) Five different four–digit integers all have the same initial digit, and their sum is divisible by four of them. Find all possible such sets of integers.

Solution:

Let $z = a + ib$. Then $z^3 = 52 - 47i$, and $\overline{z}^3 = 52 + 47i$. It follows that

$|z|^6 = 52^2 + 47^2 = 2704 + 2209 = 4913 = 17^3$ and

$a^2 + b^2 = |z|^2 = 17$.

Suppose $z = \sqrt{17}(\cos t - i \sin t)$ where $0 < t < \frac{\pi}{2}$. Then by de Moivre's theorem,

$$\cos 3t + i \sin 3t = \frac{52}{17\sqrt{17}} + \frac{47}{17\sqrt{17}} i,$$

and

$$\tan 3t = \frac{47}{52}.$$

That is,

$$\frac{3\tan t - \tan^3 t}{1 - 3\tan^2 t} = \frac{47}{52},$$

or,

$$52 \tan^3 t - 141 \tan^2 t - 156 \tan t + 47 = 0.$$

One solution is $\tan t = \frac{1}{4}$, so $\cos t = \frac{4}{\sqrt{17}}$, $\sin t = \frac{1}{\sqrt{17}}$, and

$$a + ib = z = \sqrt{17}\left(\frac{4}{\sqrt{17}} - i\frac{1}{\sqrt{17}}\right) = 4 - i.$$

Other solutions are given by

$$a + ib = z = (4 - i)\omega \text{ and } a + ib = (4 - i)\overline{\omega}$$

where ω and $\overline{\omega}$ are (complex) cube roots of unity. Thus

$$a + ib = (4 - i)(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}) = (-2 \pm \frac{\sqrt{3}}{2}) + i(\frac{1}{2} \pm 2\sqrt{3}).$$

6. Five different four–digit integers all have the same initial digit, and their sum is divisible by four of them. Find all possible such sets of integers.

Solution:

Suppose the initial digit is a. Then all five numbers lie between $1000a$ and $1000(a+1)$, and their sum lies between $5000a$ and $5000(a+1)$. Thus the four quotients are four different integers between $5000a/1000(a+1)$ and $5000(a+1)/1000a$, that is, between $5a/(a+1)$ and $5(a+1)/a$. If $a \geq 3$, there aren't four different integers in that range, so $a = 2$ or $a = 1$. If $a = 2$, the four quotients are 4, 5, 6 and 7. In that case, the sum is a multiple of 420, say $420k$, and the five numbers are

$$60k, \ \ 70k, \ \ 84k, \ \ 105k \ \text{ and } \ 101k.$$

However, for no value of k do these all lie between 2000 and 3000.

Now suppose $a = 1$. Then the four quotients lie between 5/2 and 10. There seem to be many possible sets of quotients. However, the largest of the five numbers cannot be twice as big as the smallest (they are all between 1000 and 1999 inclusive). So no quotient can be twice another (or more). This rules out any set of quotients of which 3 is the smallest. So the set of quotients is 4, 5, 6, 7, or starts with 5 or more.

If the set of four quotients has least number 5 or more, the four corresponding numbers add to at most $1/5 + 1/6 + 1/7 + 1/8 = 533/840$ of the sum, so the fifth number is at least $307/840$ of the sum, which is far larger than twice any of the others.

So, the quotients are 4, 5, 6 and 7, and for some k the numbers are, as above,

$$60k, \ \ 70k, \ \ 84k, \ \ 101k, \ \text{ and } \ 105k.$$

These all lie between 1000 and 1999 inclusive for $k = 17, 18$ and 19.

Thus, the three possible sets are $\{1020, \ 1190, \ 1398, \ 1717, \ 1785\}$,

$\{1080, \ 1260, \ 1512, \ 1818, \ 1890\}$ and $\{1140, \ 1530, \ 1596, \ 1919, \ 1995\}$.