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UNSW School Mathematics Competition 2005

Problems and Solutions

Junior Division

Problem 1. On the Island of New Monia, the natives made totem poles out of square–heads and long–heads (which were twice as tall as square–heads). The square–heads were made of mahogany, while the long–heads were made of ebony or sandalwood. The heads were stacked upright.

(i) How many different totem poles of height 5 (that is, equivalent to 5 square–heads) were possible?

(ii) How many different totem poles of height 6 were possible with a square–head at the top?

How many different totem poles of height 6 with an ebony long–head at the top?

(iii) How many totem poles of height 11 were possible?

When the Green Witchdoctors came to power, they decreed that sandalwood could no longer be used for long–heads since the sandalwood forest was under dire threat of extinction. The Witchdoctors soon realised, with the aid of the Esteemed Mathematicians, that not enough variety was possible, so they permitted square–heads to be placed either upside down or upright.

(iv) How many totem poles of height 11 were now possible?

Solution:

(i) 21

With 2 long–heads and 1 square–head there are 12 totem poles, with 1 long–head and 3 square–heads there are 8, and with 5 square–heads there is 1.

(ii) 21, 11

(iii) 1365

Let f_n be the number of totem poles of height n. Then $f_1 = 1$, $f_2 = 3$.

Also $f_n = f_{n-1} + 2f_{n-2}$ for $n \geq 3$, because totem poles of height n are either a totem pole of height $n-1$ topped with a square–head or a totem pole of height $n-2$ topped with an ebony or a sandal-wood long–head. It follows that

$$\begin{aligned}
&f_3 = 3 + 2 \times 1 = 5, \quad f_4 = 5 + 2 \times 3 = 11, \\
&f_5 = 11 + 2 \times 5 = 21, \quad f_6 = 21 + 2 \times 11 = 43, \\
&f_7 = 43 + 2 \times 21 = 85, \quad f_8 = 85 + 2 \times 43 = 171, \\
&f_9 = 171 + 2 \times 85 = 341, \quad f_{10} = 341 + 2 \times 171 = 683, \\
&f_{11} = 683 + 2 \times 341 = 1365.
\end{aligned}$$

(iv) 13860

Let g_n be the number of post–revolution totem poles of height n. Then $g_1 = 2$, $g_2 = 5$ and $g_n = 2g_{n-1} + g_{n-2}$ for $n \geq 3$. It follows that

$$\begin{aligned}
&g_3 = 12, \quad g_4 = 29, \quad g_5 = 70, \quad g_6 = 169, \quad g_7 = 408, \\
&g_8 = 985, \quad g_9 = 2378, \quad g_{10} = 5741, \quad g_{11} = 13860.
\end{aligned}$$

Problem 2.

(i) How many ways can the number 2005 be written as the sum of two squares (of integers)?

(ii) If the number 2005 is written as the sum of N different squares of positive integers, what is the largest possible value of N?

Solution:

(i) 2

$$2005 = 41^2 + 18^2 = 39^2 + 22^2.$$

(ii) 17

$$\begin{aligned}
2005 = {} & 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 \\
& + 11^2 + 12^2 + 13^2 + 14^2 + 15^2 + 18^2 + 21^2.
\end{aligned}$$

The smallest number that is the sum of 18 different squares of positive integers is

$$1^2 + 2^2 + \cdots + 18^2 = \frac{1}{6} \times 18 \times 19 \times 37 = 2109.$$

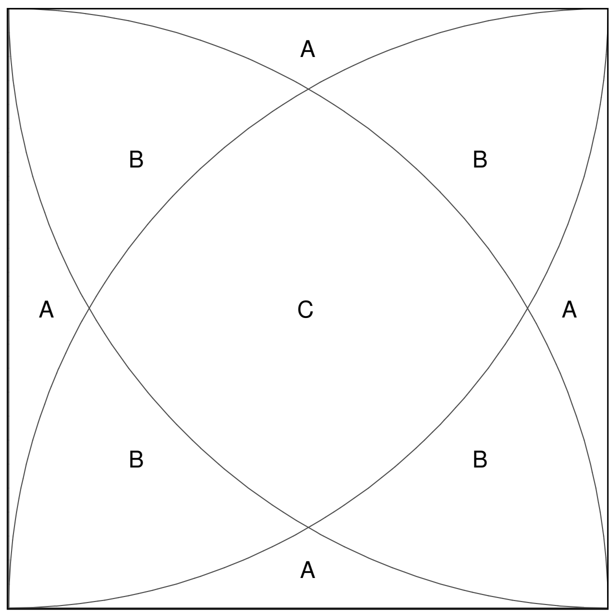
Problem 3. A disc of radius 1 unit is cut into quadrants (identical quarters), and the quadrants are placed in a square of side 1 unit.

(i) What is the least possible area of overlap?

(ii) What are the other possible areas of overlap?

Solution:

Consider the following diagram.



The possible overlaps are as follows:

If all the quadrants are tucked into the same corner of the square,

$$2A + 3B + C = \frac{\pi}{4}.$$

If the quadrants are tucked into two adjacent corners,

$$A + 2B + C = 2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + \frac{\sqrt{3}}{4} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}.$$

If the quadrants are tucked into opposite corners,

$$2B + C = 2\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{2} - 1.$$

It follows that

$$A = 1 - \frac{\pi}{6} - \frac{\sqrt{3}}{4}, \;\; B = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1, \;\; C = \frac{\pi}{3} + 1 - \sqrt{3}.$$

If the quadrants are tucked into three corners, the overlap is

$$B + C = \frac{5\pi}{12} - \frac{\sqrt{3}}{2}.$$

If the quadrants are all in different corners, the overlap is

$$C = \frac{\pi}{3} + 1 - \sqrt{3}.$$

So the answers are:
(i)

$$\frac{\pi}{3} + 1 - \sqrt{3}$$

(ii) Answers are in increasing order,

$$\frac{5\pi}{12} - \frac{\sqrt{3}}{2},\ \ \frac{\pi}{2} - 1,\ \ \frac{\pi}{3} - \frac{\sqrt{3}}{4} \text{ and } \frac{\pi}{4}.$$

Problem 4. Prove that if $a \geq b \geq c \geq 0$ then $a^2 + 3b^2 + 5c^2 \leq (a+b+c)^2$.
Can you prove a more general inequality?

Solution:

$$\begin{aligned}(a+b+c)^2 &= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc \\ &\geq a^2 + b^2 + c^2 + 2b^2 + 2c^2 + 2c^2 \\ &= a^2 + 3b^2 + 5c^2.\end{aligned}$$

More generally, if $a_1 \geq a_2 \geq \ \cdots \ \geq a_n \geq 0$,

$$\begin{aligned}(a_1 + a_2 + \ \cdots \ + a_n)^2 &= \sum_{k=1}^{n} a_k^2 + \sum_{l<m} 2a_l a_m \\ &\geq \sum_{k=1}^{n} a_k^2 + \sum_{l<m} 2a_m^2 \\ &= \sum_{k=1}^{n} a_k^2 + \sum_{m=2}^{n} (2m-2)a_m^2 \\ &= \sum_{k=1}^{n} (2k-1)a_k^2 \\ &= a_1^2 + 3a_2^2 + \ \cdots \ + (2n-1)a_n^2.\end{aligned}$$

Problem 5. A cubic box, side $1.2m$, is placed on the ground next to a high wall. A ladder, length $3.5m$, leans against the wall, and just touches the top edge of the box. How far from the box is the foot of the ladder, and how far above the box does the ladder touch the wall?

Solution:

Let the required distances be x, y. Then

$$\frac{y}{1.2} = \frac{1.2}{x},$$

or,

$$xy = 1.44. \tag{1}$$

Also

$$(x + 1.2)^2 + (y + 1.2)^2 = 3.5^2,$$

or,

$$x^2 + y^2 + 2.4(x + y) - 9.37 = 0 \tag{2}$$

Adding twice (1) to (2), we obtain the quadratic in $x + y$,

$$(x + y)^2 + 2.4(x + y) - 12.25 = 0.$$

The positive solution is

$$x + y = 2.5.$$

So now we have $x + y = 2.5$, $xy = 1.44$. It follows that x and y are the roots of the quadratic $z^2 - 2.5z + 1.44 = 0$.
It follows that $x = 1.6$, $y = 0.9$ or *vice versa*.

Problem 6. Prove that if $a > 1$ and $b > 1$ then

$$\frac{a^2}{b-1} + \frac{b^2}{a-1} \geq 8.$$

Solution:

We use the AM–GM inequality

$$\frac{x+y}{2} \geq \sqrt{xy} \text{ for } x, y \geq 0.$$

$$
\begin{aligned}
\frac{a^2}{b-1}+\frac{b^2}{a-1} &\geq 2\sqrt{\frac{a^2}{b-1}\cdot\frac{b^2}{a-1}} \\
&= 2\sqrt{\frac{a^2}{a-1}}\sqrt{\frac{b^2}{b-1}} \\
&= 2\sqrt{a+1+\frac{1}{a-1}}\sqrt{b+1+\frac{1}{b-1}} \\
&= 2\sqrt{2+\left(a-1+\frac{1}{a-1}\right)}\sqrt{2+\left(b-1+\frac{1}{b-1}\right)} \\
&\geq 2\sqrt{2+2}\sqrt{2+2} \\
&= 2\times 2\times 2 = 8.
\end{aligned}
$$

Senior Problems

Problem 1.

See Junior 3.

Problem 2.

See Junior 4.

Problem 3. The triangular numbers are given by $T_n = 1 + 2 + \cdots + n$ for n a positive integer ($T_1 = 1$).

Discover and prove a formula for

$$T_n\left(\frac{1}{T_1}+\frac{1}{T_2}+\cdots+\frac{1}{T_n}\right).$$

Hence find the sum of the reciprocals of all the triangular numbers.

Solution:

$$T_n\left(\frac{1}{T_1}+\frac{1}{T_2}+\cdots+\frac{1}{T_n}\right)=n^2,$$

or

$$\frac{1}{T_1}+\frac{1}{T_2}+\cdots+\frac{1}{T_n}=\frac{n^2}{T_n}=\frac{n^2}{(n^2+n)/2}=\frac{2n}{n+1}. \tag{1}$$

This is easily proved by induction. The inductive step depends on

$$\frac{2n}{n+1}+\frac{1}{(n+1)(n+2)/2}=\frac{2(n+1)}{(n+1)+1}.$$

It follows from (1) that

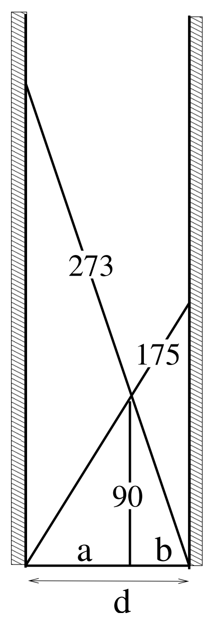
$$\sum_{n=1}^{\infty} \frac{1}{T_n} = 2.$$

Problem 4. In an alleyway with tall buildings on both sides, a ladder of length $3.9m$ leans from the foot of the west wall on to the east wall, while a ladder of length $2.5m$ leans the other way across the alleyway, from the foot of the east wall on to the west wall. Looking north along the alleyway, the ladders appear to cross $1\frac{2}{7}m$ above the roadway.

How wide is the alleyway?

Solution:

In order to avoid fractions, consider all distances as multiples of $\frac{1}{70}m$. Then we have the following diagram.



From similar triangles,

$$\frac{a}{90} = \frac{d}{\sqrt{175^2 - d^2}}, \quad \frac{b}{90} = \frac{d}{\sqrt{273^2 - d^2}}$$

so

$$a = \frac{90d}{\sqrt{175^2 - d^2}}, \quad b = \frac{90d}{\sqrt{273^2 - d^2}},$$

and

$$a + b = d.$$

It follows that

$$\frac{90d}{\sqrt{175^2 - d^2}} + \frac{90d}{\sqrt{273^2 - d^2}} = d,$$

or,

$$\frac{1}{\sqrt{175^2 - d^2}} + \frac{1}{\sqrt{273^2 - d^2}} = \frac{1}{90}.$$

It is clear from a graph that this equation has just one root. To find it, one way to proceed is as follows.

Let

$$\frac{1}{\sqrt{273^2 - d^2}} = \frac{1}{180}(1-x), \quad \frac{1}{\sqrt{175^2 - d^2}} = \frac{1}{180}(1+x)$$

where $0 < x < 1$.

Then $$273^2 - d^2 = \frac{180^2}{(1-x)^2}, \quad 175^2 - d^2 = \frac{180^2}{(1+x)^2}.$$

Then x satisfies the equation

$$\frac{180^2}{(1-x)^2} - \frac{180^2}{(1+x)^2} = 273^2 - 175^2,$$

or,

$$\frac{x}{(1-x^2)^2} = \frac{273^2 - 175^2}{4 \times 180^2} = \frac{2 \times 7^3}{45^2}.$$

This is a quartic equation for x, and has just one solution with $0 < x < 1$. If we guess that the solution is rational,

$$x = \frac{p}{q}$$

with $p,\ q$ integers, we find

$$\frac{pq^3}{(q^2 - p^2)^2} = \frac{2 \times 7^3}{45^2}.$$

And indeed, this has the solution $p = 2,\ q = 7$.

So $x = \frac{2}{7}$, and

$$d^2 = 273^2 - \frac{180^2}{(1 - \frac{2}{7})^2} = 105^2,$$

$$d = 105.$$

Dividing by 70, we find that the width of the alleyway is $1.5m$.

Problem 5. Consider the diophantine equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{2}{abc} = \frac{15}{a+b+c}.$$

(i) If $a = 3$ and $b = 5$, find c.

(ii) Show that the equation has infinitely many solutions in integers.

Solution:

(i) If $a = 3$, $b = 5$, $c = 1$ or 17.

(ii) The clue here is that if two of $a,\ b,\ c$ are specified, the third is given by a quadratic.
From $b = 5,\ c = 17$ we find $a = 3$ or 29.
From $c = 17,\ a = 29$ we find $b = 5$ or 99.
From $a = 29,\ b = 99$ we find $c = 17$ or 169.
We obtain the sequence

$$1,\ 3,\ 5,\ 17,\ 29,\ 99,\ 169,\ \cdots .$$

Observe the two sets $1,\ 3,\ 5$ and $5,\ 17,\ 29$ in arithmetic progression. If we set $a = p - q,\ b = p,\ c = p + q$ the equation becomes

$$\frac{1}{p-q} + \frac{1}{p} + \frac{1}{p+q} + \frac{2}{p(p^2-q^2)} = \frac{15}{3p},$$

or, after some manipulation,

$$p^2 - 2q^2 = 1.$$

Now, it is well–known that this equation, known as Pell's equation, has infinitely many solutions in integers, given by

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \begin{pmatrix} 3 \\ 2 \end{pmatrix},\ \begin{pmatrix} 17 \\ 12 \end{pmatrix},\ \begin{pmatrix} 99 \\ 70 \end{pmatrix},\ \cdots$$

and so on. These pairs are given by

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = 6\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix} - \begin{pmatrix} p_{n-2} \\ q_{n-2} \end{pmatrix}$$

or

$$\begin{aligned} p_n &= 3p_{n-1} + 4q_{n-1}, \\ q_n &= 2p_{n-1} + 3q_{n-1}. \end{aligned}$$

The pairs $\begin{pmatrix} p_n \\ q_n \end{pmatrix}$ turn up among the convergents to the continued fraction for $\sqrt{2}$:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

The convergents are

$$\frac{1}{1},\ \frac{\mathbf{3}}{\mathbf{2}},\ \frac{7}{5},\ \frac{\mathbf{17}}{\mathbf{12}},\ \frac{41}{29},\ \frac{\mathbf{99}}{\mathbf{70}},\ \cdots .$$

But note that the full sequence

$$1,\ 3,\ 5,\ 17,\ 29,\ 99,\ \cdots$$

also appears in the sequence of convergents, alternately as denominators and numerators:

$$\frac{1}{\mathbf{1}},\ \frac{\mathbf{3}}{2},\ \frac{7}{\mathbf{5}},\ \frac{\mathbf{17}}{12},\ \frac{41}{\mathbf{29}},\ \frac{\mathbf{99}}{70},\ \frac{239}{\mathbf{169}},\ \cdots .$$

The rule is $x_1 = 1$, $x_2 = 3$ and for $n \geq 3$,

$$x_n = \begin{cases} 2x_{n-1} - x_{n-2} & \text{for } n \text{ odd} \\ 4x_{n-1} - x_{n-2} & \text{for } n \text{ even.} \end{cases}$$

Of course, everything that I have claimed needs proof, but the proofs are omitted for the time being.

Problem 6.

(i) Two people play a game by taking turns to toss a coin. The first to throw both Heads and Tails wins. What is the probability the first to throw wins?

(ii) What are the probabilities with three people playing?

What are the probabilities with n players?

(iii) Two people play a game by taking turns to spin a spinner with three equally likely outcomes. The first to obtain all three results wins. What is the probability the starter wins?

Solution:

(i)

$$\Pr\{A \text{ wins}\} = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \cdots = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3},$$

$$\Pr\{B \text{ wins}\} = \frac{1}{3}.$$

(ii)

$$\Pr\{A \text{ wins}\} = \frac{1}{2} + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^7 + \cdots = \frac{4}{7},$$

$$\Pr\{B \text{ wins}\} = \frac{2}{7},$$

$$\Pr\{C \text{ wins}\} = \frac{1}{7}.$$

In an n person game,

$$\Pr\{A_k \text{ wins}\} = \frac{2^{n-k}}{2^n - 1}.$$

(iii) The probability that the starter wins is $\frac{81}{140}$.

Let $P_1(x_1, x_2)$ be the probability that the player who is about to spin wins given that their score is x_1 and the other person's score is x_2, and let $P_2(x_1, x_2)$ be the probability that the other person wins.

Then
$$P_1(2,2) = \frac{1}{3} + \frac{2}{3}P_2(2,2),$$
$$P_2(2,2) = \frac{2}{3}P_1(2,2).$$

It follows that
$$P_1(2,2) = \frac{3}{5}, \quad P_2(2,2) = \frac{2}{5}.$$

Also
$$P_1(2,1) = \frac{1}{3} + \frac{2}{3}P_2(1,2),$$
$$P_2(2,1) = \frac{2}{3}P_1(1,2),$$
$$P_1(1,2) = \frac{1}{3}P_2(2,1) + \frac{2}{3}P_2(2,2),$$
$$P_2(1,2) = \frac{1}{3}P_1(2,1) + \frac{2}{3}P_1(2,2).$$

It follows that
$$P_1(2,1) = \frac{27}{35}, \quad P_2(2,1) = \frac{8}{35}, \quad P_1(1,2) = \frac{12}{35}, \quad P_2(1,2) = \frac{23}{35}.$$

Finally,
$$P_1(1,1) = \frac{1}{3}P_2(1,1) + \frac{2}{3}P_2(1,2),$$
$$P_2(1,1) = \frac{1}{3}P_1(1,1) + \frac{2}{3}P_1(1,2).$$

It follows that
$$P_1(1,1) = \frac{81}{140}, \quad P_2(1,1) = \frac{59}{140}.$$