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History of Mathematics: Logic Problems

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Sometimes in these columns I will allow myself the liberty of straying outside the domain of History, properly so called, in order to look at somewhat wider issues. This is what I am doing here, but to set the scene I will begin with a reminiscence. In the years shortly following World War II, a now vanished Melbourne newspaper, *The Argus*, carried a much-praised education column, and for each of six years they collected much of its material into an Annual entitled *The Argus Students' Practical Notebook*. The 5th such volume (1952) was my own introduction to a puzzle which, in one form or another, is quite famous. I will stick to their version:

> To demonstrate the intelligence of his friend Hermon Wizzer, Fishwit Herring once arranged this little problem.
>
> He showed Wizzer, Merrick and Veale five caps, three of which were blue and two red. Then he sat them down one in front of the other so Veale could see Merrick and Wizzer, Merrick could see Wizzer and Wizzer could not see anyone.
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> Then Fishwit put a cap on the head of each of his three friends, after which he asked Veale if he knew the color of the cap he was wearing. Veale said 'No'. Merrick was also unable to name the color of his cap. But when Wizzer was asked, he correctly named the color of his own cap, although from his position he could not see any of the caps.
>
> *Now the question is this: How did Wizzer give the correct answer, and what was that answer? He did not guess, nor did he have the use of a mirror.*

The answer given by *The Argus* goes like this. If Merrick and Wizzer had both had red caps, then Veale would have deduced that his own cap was blue (because there were only two red caps). From this it follows that at least one of either Merrick or Wizzer had on a blue cap. Wizzer made this deduction from Veale's response. We then come to Merrick's inability to deduce the color of his cap. Had Merrick seen a red cap on Wizzer's head, then he (Merrick) would have been able to deduce that the color of his own was blue. This he could not do. Thus Wizzer's cap was blue.

This is the standard form in which this problem is posed. First notice that, although we now know that Wizzer's cap is blue, we still do not know the color of the other two. If Merrick's cap is blue, then Veale's may be either red or blue, and all the conditions of the problem are met. (Veale sees two blue caps.) On the other hand, if Merrick's cap is red, then Veale's may still be either red or blue. (Veale sees one red cap and one blue.)

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Thus what we have is a situation in which Wizzer has a blue cap and the other caps may be arranged perfectly at random.

This problem is an example of a type collectively referred to as 'logic problems'. In the present context, the name strikes me as something of a misnomer. Strictly speaking, Logic refers to the secure deduction of a conclusion from a set of premises.

Here we are in a somewhat different realm. Wizzer's conclusion rests on the supposed deductive powers of Merrick and Veale. Assuming that both of these answered truthfully, we can be confident that Veale would have to be pretty thick to see two red hats and still answer as he did. Merrick's deduction is rather more difficult: still relatively easy, but no longer trivially so. Wizzer has to have complete confidence in Merrick's cognitive abilities if he is to be certain of his own answer! There are other 'logic problems' that do not have this character, but here I am not concerned with these.

Back to the example — just suppose that in fact Wizzer had had on a red cap, and Merrick a blue, then Merrick should have been able to know this, but if he failed to draw the correct conclusion, then Wizzer would have been led into error.

Now consider Herring's role in all this. His demonstration of Wizzer's intelligence can only work if he puts a blue cap on Wizzer's head; had he used a red one then either Merrick or Veale would have been able to give the correct color of his cap. This holds whatever caps he puts on the other heads, as readers may check. Because Herring wanted 'demonstrate the intelligence of his friend Hermon Wizzer', he would have to have prepared this aspect of his 'demonstration' in advance. Moreover, the intelligent Wizzer would have known this, and hence he need not have paid any attention to the replies given by Veale and Merrick. Even discounting the possibility that he colluded with Herring, he would have known from the outset that his own cap was blue!

Here we are attributing *motive* to Herring, and this is another aspect of many 'logic problems' that strays outside strict Logic.

In practice, such argumentation works, but not all the time. My own favorite example concerns the decision I have to make as I approach a set of traffic lights. Suppose that my intention is to proceed straight ahead. Other things being equal, I do best to stay in the right lane (the 'fast lane'), but if I find myself stuck behind another motorist, who wants to turn right, then I lose out as all the motorists in the left lane stream past me.

However, I reason as follows: if the car at the front (which I am not in a position to observe directly) is indicating an intention to turn right, then the car immediately behind that would not take up this position unless also wanting to turn, in which case, it would so indicate; and so on. The car at the back of the queue (the one I can see), if not indicating its intention to turn, sees no indication in front of it, and this means that the car before that one is not so indicating, etc.

Such chains of reasoning are referred to as 'inductive'. The argument comprises a sequence of conclusions each depending on the previous one. If each step of the way were entirely logical, then the end conclusion would be quite guaranteed. In practice, as the example of the right-hand turn makes clear, it is not. In that case, there is always the possibility that one of the drivers between me and the front of the queue will have failed to draw the conclusion that a delay would occur, might not have been concerned about it, or may have been deceived by the car in front suddenly (and maddeningly) changing its intended direction.

If the queue is long enough, a probabilistic factor comes into play, and in such cases, I tend to stay in the left lane. If the probability of each motorist behind the first in a queue n vehicles long arriving at the correct decision is $1-\epsilon$, then the chance that all of them have been sensible is $(1-\epsilon)^{n-1}$. My chance of being delayed is then $1-(1-\epsilon)^{n-1}$, and even when ϵ is small, this chance can be unacceptably large if n is also large. [If $\epsilon = 0.1,$ my chances of proceeding unimpeded are less than 0.5, once $n > 7$.]

Something of this line of thought takes hold of me when I see some of the more outlandish 'logic problems'. One that, in one form or another, keeps resurfacing, goes by many names. I first heard of it as 'the three women with sooty faces'. More usually, the statements are endowed with lurid plots: one bizarre version is known as the '40 monks problem' (this variant was discussed by Peter Grossman in *Function*, August 1995). Another has recently appeared in the August 2006 issue of the *Gazette of the Australian Mathematical Society* with 100 superstitious tribesmen. The version I shall analyze is that to be found on the web at:

http://en.wikipedia.org/wiki/Induction_Puzzles

There it is given the name 'Josephine's Problem' and it goes as follows:

> In Josephine's Kingdom every woman has to take a logic exam before being allowed to marry. Every married woman knows about the fidelity of every man in the Kingdom *except* for her own husband, and etiquette demands that no woman should tell another about the fidelity of her husband. A gunshot fired in any house in the Kingdom will be heard in any other. Queen Josephine announced that unfaithful men had been discovered in the Kingdom, and that any woman knowing her husband to be unfaithful was required to shoot him at midnight following the day she discovered his infidelity. How did the wives manage this?

Here is the published solution:

> If there is only 1 unfaithful husband, then every woman in the Kingdom knows that, except for his wife, who believes that everyone is faithful. Thus, as soon as she hears from the Queen that unfaithful men exist, she knows her husband must be unfaithful, and shoots him.
>
> If there are 2 unfaithful husbands, then both their wives believe there is only 1 unfaithful husband (the other one). Thus, they will expect that the case above will apply, and that the other husband's wife will shoot him at midnight on the next day. When no gunshot is heard, they will realize that the case above did *not* apply, thus there must be more than 1 unfaithful husband and (since they know that everyone else is faithful) the extra one must be their own husband.
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> If there are 3 unfaithful husbands, each of their wives believes there to be only 2, so they will expect that the case above will apply and both husbands

will be shot on the second day. When they hear no gunshot, they will realize that the case above did *not* apply, thus there must be more than 2 unfaithful husbands and as before their own husband is the only candidate to be the extra one.

In general, if there are n unfaithful husbands, each of their wives will believe there to be $n-1$ and will expect to hear a gunshot at midnight on the $(n-1)$th day. When they don't, they know their own husband was the nth.

When n gets large, this simply defies credibility. With $n = 3$ (the ladies with sooty faces), it may be plausible, but when $n = 40$ (the 40 monks) it is clearly not. A recent account of the puzzle (by Norman Do, in the *Gazette of the Australian Mathematical Society*, Vol. 33 (2006), p. 157) has 100 religious fanatics, of whom k simultaneously suicide on the kth day, and the remaining $100 - k$ likewise kill themselves on day $k + 1$. This analysis holds even when the number 100 is replaced by an arbitrary n, however large.

Do's analysis points to an interesting aspect of the puzzle. Suppose in the 'unfaithful husbands' version of the problem, that Josephine had merely informed each wife, privately and individually, that she knew of the existence of unfaithful husbands. In this case, each wife would know that unfaithful husbands were among them in the kingdom, but she would not know that *each of the other wives also knew this*. Hence it would not be possible for the wronged wives to make the deduction they are supposed to have made. (If there were one unfaithful husband, he would die, but if there were 2 or more, then all would survive!) Important to the analysis is the aspect that not only do the wives know what they know, but they also know that every other wife is in possession of the same information.

Basic to the posing of such problems is what is known as the *theory of mind.* Early in our mental development, we acquire a sense of our own identity and come to realize that other people are similarly 'selves', who think in much the same way as we do. This second realization is the theory of mind.

(Modern psychological research sees the development of a theory of mind as crucial to the acquisition of our capacity to reason. Two excellent books on these researches are Julian Keenan's *The Face in the Mirror* (Harper Collins, 2003) and Peter Hobson's *The Cradle of Thought* (Pan, 2004). The human ability to reason symbolically, according to Hobson, has its roots in play, pretense and mother-child interaction. Failure to develop a full theory of mind results, in its most severe manifestation, in the condition known as *autism*.)

The human thought that develops along this pathway is different from the artificial 'thought' that a computer produces. The lengthy inductive chains of reasoning are more typical of computer responses than of human ones. The human response is prone to error, in ways that computers are not. Humans can also become bored, inattentive or overwhelmed, in situations where computers do not.

Strict Logic is closer to the model of computer thought than to real human thought. One aspect of this distinction is the distinction between the validity of an argument and the truth of the conclusion. From a false premise, anything can be made to follow.

The deduction that follows from the premises is not concerned at all with the status of the premises on which the subsequent reasoning depends.

This was nicely, and somewhat quirkily, illustrated by Lewis Carroll (in real life the mathematician C. L. Dodgson) in his book *Curiosa Mathematica*. (The reference is to the English nursery rhyme *Mistress Bond*.)

> 'I have sent for you, my dear Ducks', said the worthy Mrs. Bond, 'to enquire with what sauce you would like to be eaten?' 'But we don't want to be killed', cried the Ducks. '*You are wandering from the subject*," was Mrs Bond's perfectly logical reply.

Mrs Bond has introduced a premise that the ducks find not at all to their liking, and her subsequent discourse depends entirely on the acceptance of this. It is a ploy widely used in debate, e.g. in the political sphere. Perhaps the most notorious example in recent Australian history was the referendum conducted in Tasmania on the damming of the Franklin River. The 'No dams' option favored by some 50% of the voters was simply not presented!

In such cases, the question of motive, another aspect of human intelligence, becomes important. We saw it briefly in the 'demonstration' by Herring of Wizzer's intellectual abilities. It also colors the way in which such problems are presented. Such considerations matter particularly in probabilistic questions.

One famous such problem is best known as the 'Monty Hall Problem'. It goes like this. Monty Hall, a U. S. TV game-show host, presents a competitor with three closed doors and asks for one to be chosen. The competitor will receive as a prize whatever is hidden behind the door chosen. One hides an expensive car (which the competitor is presumed to want), the others goats (presumed unwanted). The competitor chooses (say) Door No. 1. Hall then opens Door No. 3 to reveal a goat, and invites the competitor to change the original choice and pick instead Door No. 2. Should the competitor do so?

The problem has generated a lot of heated controversy, and the best analysis, as far as it goes, is a discussion by Leonard Gillman in *American Mathematical Monthly* (January 1992). He set q as the probability that Hall would open Door No. 3 in the event that the car was actually behind Door No. 1 (in which case, he would not, we are to presume, open Door No. 1). Now set P as the probability that the car is actually behind Door No. 2. Gillman calculated that $P = \frac{1}{1+q}$. However, the value of q is unknown; it could be any number between 0 and 1. At one extreme, if $q = 0$, then $P = 1$, and the car is definitely behind Door No. 2; at the other extreme, if $q = 1$, then $P = \frac{1}{2}$ and one neither gains nor loses by switching. It is not difficult to show that in all cases $\frac{1}{2} \leq q \leq 1$, and so the competitor increases the chance of success by switching and choosing Door No. 2.

When I read Gillman's paper, however, I thought it incomplete. Put yourself in the competitor's place. Hall's behavior, after you have chosen Door No. 1, comes as a complete surprise. 'What are his motives?' you ask yourself. Perhaps he is trying to lead you astray, or perhaps he is trying to do you a favor. You might suppose that the probability that Hall's intentions are malevolent is p, and that they are the opposite are $1 - p$. Now, although we have no access to the possible value of q, we might hope to estimate p (by close scrutiny of Hall's demeanor, etc). This changes our assessment of the situation.

One way to proceed is to ignore the previous analysis and run simply with your intuition, i.e. change your choice if you judge that $p < \frac{1}{2}$. This, however, is to disregard the conclusions of Gillman's analysis. Here is a different, and better, way to proceed.

On the face of it, there are four possibilities:

1. The car is behind Door No. 2 and Hall is benevolent
2. The car is behind Door No. 2 and Hall is malevolent
3. The car is behind Door No. 1 and Hall is benevolent
4. The car is behind Door No. 1 and Hall is malevolent

However, of these four, possibilities 2 and 3 are impossibilities; they involve self-contradictory assumptions. We are thus constrained to consider only Possibilities 1 and 4. On the face of this, the probability of Possibility 1 is $P(1-p)$ and that of Possibility 4 is $(1-P)p$. However, these 'probabilities' do not add up to 1, but rather to $P+p-2Pp$, i.e. to $\frac{1-p+pq}{1+q}$. So, in order to arrive at true probabilities, we need to divide both expressions by this amount. (This is essentially Bayes' Theorem.)

The probability of Possibility 1 is thus $\frac{1-p}{1+pq-p}$. This becomes the probability of the contestant's winning the car. In this formula, if $p = \frac{1}{2}$, Hall is completely neutral, and Gillman's result is returned. On the other hand, because we have no way of knowing q, we might choose the case $q = \frac{1}{2}$, which then gives the probability of success as $\frac{1-p}{1-p/2}$. For a better than even chance of winning the car, we need $\frac{1-p}{1-p/2} > \frac{1}{2}$ that is to say $p < \frac{2}{3}$. So if we are to assume that Hall actually picks Door No. 3 at random, then we need to believe that the likelihood of his being 'against us' is less than $\frac{2}{3}$, or in other words, the probability that he is trying to do us a favor is greater than $\frac{1}{3}$.

Questions of probability are particularly prone to ambiguity of interpretation. Part of the reason for this is that we can, even without knowing it, import an assumption that is (strictly speaking) not stated as part of the problem. I can illustrate this by referring to an event I actually witnessed. An eminent probabilist was delivering a schools lecture on his subject. At one point, he tossed two coins, covered them with his hands and asked for the probability of 'two tails' given the information that 'one of the coins is showing tails'.

This is in fact a straightforward application of Bayes' Theorem. A *priori*, there are four possibilities: HH, HT, TH and TT (with the obvious notation). However, the information provided rules out the first of these, leaving only three possibilities. Of these three, one displays the required character ('two tails'), and so the answer to the question posed is $\frac{1}{3}$.

However, the lecturer, who posed the question several times, said each time 'one of the coins is showing tails' and glanced down toward the right-hand coin. This may well have been a quite unconscious gesture, but it changes the nature of the problem. If we interpret the question as asking for the probability that the two coins both show tails, given that a *particular* one (that on the right) is already known to be showing tails, then the probability that both are doing so becomes $\frac{1}{2}$, the answer in fact given by most of the audience responses.

The interesting point is that it may well be that the audience picked up on the lecturer's gesture without themselves realizing it, and so gave an answer to a problem somewhat different from the one posed by the words alone.

This effect is known and it is guarded against in routine statistical trials. Suppose that a new drug treatment is being trialed. It is tested on a sample of patients, but only half of the patients receive the drug under test. The others are given an inactive 'placebo'. The nurses actually delivering the drug are kept quite unaware of which patients are in which group, as also of course are the patients whose progress is under review. This procedure is thus referred to as 'double blind' and it is adopted to prevent the subtle transmission from nurse to patient by 'body language' and suchlike cues.

But we do not require a statistical context to encounter contexts in which 'logic problems' depend critically on slight nuances of possible meaning. Perhaps the most notorious is the so-called 'impossible problem'. It was given this name by the *Scientific American* columnist Martin Gardner, whose own discussions of the matter were uncharacteristically sloppy and marred by error.

There are various forms of it, and the differences between them make for different possible answers. Here is one version:

Let x, y ('the numbers') be positive integers such that $2 \leq x \leq y$, and let $P = xy$ and $S = x + y$. Two discussants, A and B have partial information on the values of x and y in that A knows the value of P and B knows that of S. The following dialogue now takes place:

A:	I do not know the numbers.	[Statement 1]
B:	I could have told you so!	[Statement 2]
A:	Now I do know the numbers.	[Statement 3]
B:	So do I.	[Statement 4]

In this form, the problem of determining the numbers from a knowledge of this dialogue appeared in *Function* (April 1977) and earlier in *Mathematics Magazine* (March 1976). However there were subtle differences between the two versions in that the *Function* version stipulated that $y \leq 200$, while the other had $S \leq 100$.

[Martin Gardner included the problem (with a somewhat streamlined version of the same dialogue) in his column for December, 1979. However, Gardner stipulated the bound $y \leq 20$, which gives a version having no solution. Gardner claimed the solution $\{x, y\} = \{4, 13\}$. However, in that case, $P = 52$, and A could deduce that either $\{x, y\} = \{4, 13\}$ or $\{x, y\} = \{2, 26\}$. Then, knowing that $y \leq 20$, A could deduce the first possibility and would not make Statement 1. Gardner retracted his original statement in March, 1980, and printed further comment in May, 1980. The trouble is that he kept referring to 'the upper bound' without making it clear whether S or y was being bounded. This makes a big difference! There are also other difficulties with his various accounts.]

The *Function* version was solved by Christopher Stuart, then an undergraduate at the University of Melbourne, and his solution was printed in the issue for February, 1978. Stuart began by using the bound on y to determine first that $S < 103$, and later, by means of a more subtle argument that $S < 51$. Thus, the *Function* version actually explored less of the $\{x, y\}$-space than did the one in *Mathematics Magazine*. I think myself that the bound on S makes for a much more satisfactory problem, and it is certainly true that the *Mathematics Magazine* solution gives a stronger result than that in *Function*. The only use made of the bound (on S) is to restrict the search.

The solution published there was sent in by a problem-solving group from Bern (Switzerland). They began by noting that if P can be expressed either as the product of two primes, or else as a cube of a prime, then A knows $\{x, y\}$ immediately, and so does not make Statement 1. But then Statement 2 tells us that B has enough knowledge to see that neither of these situations is possible. Now if S is even, it *could* be the sum of two primes. [All *even* numbers less than 3×10^{17} are known to be expressible as the sum of two primes. A major unsolved problem (known as the Goldbach Conjecture) is that all primes are so expressible. Certainly the result holds for all even numbers less than 100.] So, if S is even, then it might well be the sum of two (odd) primes, and so Statement 2 could not be made. The sum of a prime and its square is always even also. We thus know that S is odd. We can also rule out other values of S, because if S is the sum of an odd prime and 2, then again, Statement 2 cannot be made.

Thus the Bern group came up with a set $\mathbf{M}$ comprising a list of odd numbers (greater than 3 and less than 100) omitting those (such as 5) which are of the form $p + 2$, where p stands for a prime. Statement 2 says, in translation: $S \in \mathbf{M}$. The set $\mathbf{M}$ has 24 members. The group then proceeded to reduce this number further. They noted that if S may be expressed in the form $S = 2^n + p$, then A, having heard Statement 2, will know that $P(= 2^n p) \Rightarrow \{x, y\} = \{2^n, p\}$ and thus can make Statement 3. However, if two such partitions of S are possible, then B will not know whether the one or the other has been used, and so cannot make Statement 4.

The case $S = 11$ is instructive and typical. $11 = 4+7 = 8+3$. If $P = 28$, then $\{x, y\} = \{4, 7\}$, but if $P = 24$, then $\{x, y\} = \{3, 8\}$, and B will not know which case applies. This consideration reduces the set $\mathbf{M}$ to a smaller set $\mathbf{L}$ which has only 8 members: $\{17, 29, 41, 53, 59, 65, 89, 97\}$. These are dealt with using more subtle considerations.

The case $S = 65$ is instructive, reasonably typical and needed for what follows. $65 = 4 + 61 = 32 + 33$. If the first possibility, then $P = 244$, and A, knowing that S is odd, will rule out the possibility $\{x, y\} = \{2, 122\}$ and deduce $\{x, y\} = \{4, 61\}$, while if $P = 1056$, A will use a similar logic to rule out all possibilities except $\{x, y\} = \{32, 33\}$, but B will be unable to determine which of these deductions A has made.

By means of similar arguments, they then eliminate a further six possibilities: $\{29, 41, 53, 59, 89, 87\}$, thus leaving only the possibility $S = 17$. In order to establish that this possibility in fact works, they examine every partition of 17. We have:

1. $17 = 2 + 15 \Rightarrow P = 30,$
2. $17 = 3 + 14 \Rightarrow P = 42,$
3. $17 = 4 + 13 \Rightarrow P = 52,$
4. $17 = 5 + 12 \Rightarrow P = 60,$
5. $17 = 6 + 11 \Rightarrow P = 66,$
6. $17 = 7 + 10 \Rightarrow P = 70,$
7. $17 = 8 + 9 \Rightarrow P = 72.$

Cases 1, 2 and 4-7 all fail to exhibit unique factorization and hence Statement 3 cannot be made. Only Case 3 remains, and once A asserts Statement 3, B knows $\{x, y\}$.

This problem indicates the extreme limit of the 'logic puzzle'. It is highly unlikely that two humans could carry out the numerous computations required. Attempts to extend the result all depend on using computers. Computers are much better than humans where such problems arise. They may lack a 'theory of mind', but they don't get bored, and their memories are more reliable!