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The Two Pillars of Metrical Geometry

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There are *two really fundamental theorems in metrical geometry*. One of them you already know—it is *Pythagoras' theorem*. The other one is the *Triple quad formula,* which you probably don't know. These two theorems are properly the cornerstones of Euclidean geometry, spherical geometry and hyperbolic geometry, and once you understand them in the right way, *trigonometry becomes much simpler*.

This paper shows you how these two theorems flow naturally from Euclid's geometry, and lead easily to the main laws of *rational trigonometry*. This theory was introduced in 2005 in [2], see also [3].

Pythagoras' Theorem

According to the ancient Greeks, *area*—not *distance*—is the fundamental measurement in planar geometry. Area is an affine quantity, in the sense that under dilations, shears and other linear transformations, the ratios of areas are preserved. To measure a line segment the Greeks constructed a square on it, and determined the area of that square—a process called *quadrature*.

Note how different this point of view is to the one taught these days in schools, where the area of a rectangle is introduced as the product of two distance measurements. The Greeks considered things the other way around.

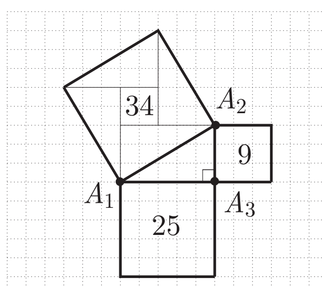

Figure 1: Pythagoras' theorem: $9 + 25 = 34$

Let's illustrate this with a Cartesian point of view, where the plane is modelled on a sheet of graph paper divided into equal cells by equally spaced horizontal and vertical lines. For simple figures, measuring area amounts to counting cells. *Parallel* and *perpendicular* lines can also be defined easily: a line with direction $(3, 2)$ is parallel to a line with direction $(6, 4)$, and perpendicular to a line with direction $(-2, 3)$. More

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generally, a line with direction (a, b) is parallel to a line with direction (c, d) precisely when $ad - bc = 0$, and is perpendicular to that line precisely when $ac + bd = 0$.

Pythagoras' theorem is a relation concerning the areas of the three squares built on each of the sides of a *right triangle*—a triangle in which two of the sides are perpendicular. Figure 1 shows such a triangle with smaller squares of areas 9 and 25. The area of the larger square can also be easily determined: subdivide it into a smaller 2×2 square and four right triangles which form two 3×5 rectangles, for a total of $4 + 2 \times 3 \times 5 = 34$. So in this case Pythagoras' theorem amounts to $9 + 25 = 34$, a result established by *counting*. This case is however somewhat special, since two of the sides of the triangle are already along the grid directions.

Figure 2 shows a more general situation in which the sides of the right triangle are not lined up with the coordinate directions. Again you can check by simple rearranging that the three squares have areas 10, 40 and 50. And indeed $10 + 40 = 50$.

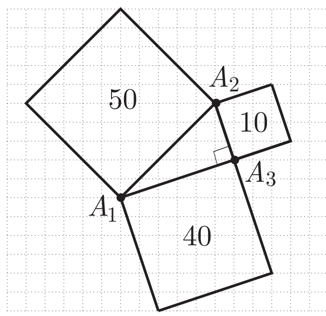

Figure 2: Pythagoras' theorem: $10 + 40 = 50$

Motivated by the idea of quadrature, we define the **quadrance** Q of a line segment to be the area of a square constructed on it. To be more precise, if a line segment is given by a vector $\overrightarrow{v} = (a, b)$ then we define the quadrance to be the area of the square formed by $\overrightarrow{v}$ and $B\left(\overrightarrow{v}\right) = (-b, a)$. Either by rearranging and counting, or by the simple determinantal formula

$$\det \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a^2 + b^2$$

we conclude the quadrance to be

$$Q = a^2 + b^2.$$

If A_1 and A_2 are two points, then $Q(A_1, A_2)$ denotes the quadrance of the line segment between them. Then the *true* Pythagoras' theorem has the following form:

Theorem 0.1 (Pythagoras) *The sides $\overline{A_1A_3}$ and $\overline{A_2A_3}$ of the triangle $\overline{A_1A_2A_3}$ are perpendicular precisely when the quadrances $Q_1 = Q(A_2, A_3)$, $Q_2 = Q(A_1, A_3)$ and $Q_3 = Q(A_1, A_2)$ satisfy*

$$Q_1 + Q_2 = Q_3.$$

Some of the advantages of this ancient Greek formulation of the theorem are obvious, others are more subtle. The main benefit is easy for everyone to understand—it deals with *rational numbers*, not their *irrational square roots*. Calculating a square root by hand is difficult, and the final result is typically only an approximation. For example, here is the beginning of the decimal expansion of a well-known square root:

$$\sqrt{5} = 2.\,236\,067\,977\,499\,789\,696\,409\,173\,668\,731\,276\,235\,440\,618\,35\ldots.$$

There are deep mysteries, as well as difficulties, contained in such an infinite decimal expansion. Using quadrance instead of distance often allows us to be more accurate. In addition, the purely algebraic form of Pythagoras' theorem means that it extends to arbitrary number fields, for example finite fields, or the complex numbers. We'll see in another paper that it also extends to Einstein's relativistic geometry.

How do we prove Pythagoras' theorem? Suppose that $\overrightarrow{A_1A_3} = (a,b)$ and $\overrightarrow{A_3A_2} = (c,d)$, so that $\overrightarrow{A_1A_2} = (a+c, b+d)$. Then the condition $Q_1 + Q_2 = Q_3$ amounts to the equation

$$c^2 + d^2 + a^2 + b^2 = (a+c)^2 + (b+d)^2.$$

But this is equivalent to

$$0 = 2\,(ac + bd)$$

which after a division by 2 is exactly the condition of perpendicularity between $\overrightarrow{A_1A_3}$ and $\overrightarrow{A_3A_2}$.

Now let us turn to that *other* pillar of geometry.

The Triple Quad Formula

The configuration when the three points A_1, A_2 and A_3 are collinear also plays a special role. In this case also there is an important relation between the three quadrances $Q_1 = Q(A_2, A_3)$, $Q_2 = Q(A_1, A_3)$ and $Q_3 = Q(A_1, A_2)$ but it turns out to be more complicated algebraically. Perhaps for this reason Euclid did not discover it, and so it is much less well known. Figure 3 shows three collinear points and the three squares determined by them. It is again an easy exercise to check that these areas are $Q_3 = 40$, $Q_2 = 90$ and $Q_3 = 10$.

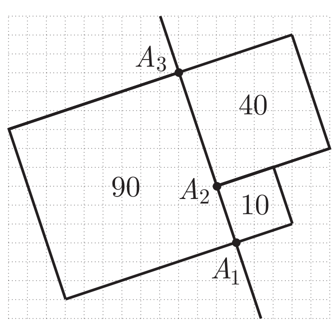

Figure 3: Triple Quad Formula: $(10 + 40 + 90)^2 = 2\,(10^2 + 40^2 + 90^2)$

Theorem 0.2 (Triple Quad Formula) *The three points A_1, A_2 and A_3 are collinear precisely when the quadrances $Q_1 = Q(A_2, A_3)$, $Q_2 = Q(A_1, A_3)$ and $Q_3 = Q(A_1, A_2)$ satisfy*

$$(Q_1 + Q_2 + Q_3)^2 = 2\left(Q_1^2 + Q_2^2 + Q_3^2\right).$$

This is rather more complicated than Pythagoras' theorem, and it is helpful to write down a simpler but less symmetrical version:

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2. \tag{1}$$

Please check, by expanding out both expressions, that they are equivalent. For the example in Figure 3 you can check that $(10 + 40 - 90)^2 = 4 \times 10 \times 40$.

How do we prove the Triple quad formula? Suppose that $\overrightarrow{A_1A_3} = (a, b)$ and $\overrightarrow{A_3A_2} = (c, d)$, so that $\overrightarrow{A_1A_2} = (a + c, b + d)$. Then the condition (1) amounts to the equation

$$\left(c^2 + d^2 + a^2 + b^2 - (a + c)^2 + (b + d)^2\right)^2 = 4\left(c^2 + d^2\right)\left(a^2 + b^2\right)$$

But this is equivalent to

$$(ac + bd)^2 = \left(c^2 + d^2\right)\left(a^2 + b^2\right).$$

The famous identity of Fibonacci

$$(ac + bd)^2 + (ad - bc)^2 = \left(c^2 + d^2\right)\left(a^2 + b^2\right)$$

then shows that our condition is equivalent to

$$ad - bc = 0$$

which is exactly the condition that the vectors $\overrightarrow{A_1A_3}$ and $\overrightarrow{A_3A_2}$ are parallel, that is that A_1, A_2 and A_3 are collinear.

The proof in fact establishes a stronger result, once we recognize that the determinantal quantity $ad - bc$ is, up to sign, twice the area a of the triangle $\overline{A_1A_2A_3}$. An alternate form in terms of the lengths of the sides is usually called Heron's formula, but it is clear from Arab sources that Archimedes knew this result.

Theorem 0.3 (Archimedes' theorem) *The area a of the triangle $\overline{A_1A_2A_3}$ is given in terms of the quadrances $Q_1 = Q(A_2, A_3)$, $Q_2 = Q(A_1, A_3)$ and $Q_3 = Q(A_1, A_2)$ by the formula*

$$16a^2 = (Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right).$$

Spread between lines

The key idea behind rational trigonometry is to substitute the notion of *spread* for the classical notion of *angle*. There are many advantages—a cleaner more logical development, more accurate calculations, generalizations to relativistic geometries and to other fields, and freedom from transcendental circular functions when dealing with geometry involving triangles. Angles are needed for *uniform motion around a circle*, which is mechanics, and a more advanced subject than trigonometry—although modern usage tends to lump both subjects together.

The **spread** $s(l_1, l_2)$ between two lines in the plane may be defined in terms of quadrance as follows. Suppose the lines meet at a point A, and B is any other point on either of the two lines, with C the foot of the perpendicular from B to the other line as in the figure. Then

$$s(l_1, l_2) = s = \frac{Q(B,C)}{Q(A,B)} = \frac{Q}{P}.$$

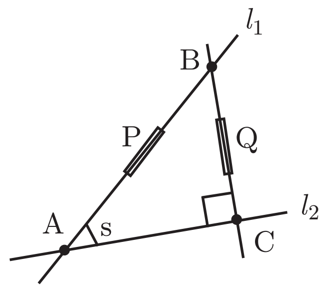

Figure 4: Spread: $s = Q/P$

For lines l_1 and l_2 with equations $a_1x + b_1y + c_1 = 0$
and $a_2x + b_2y + c_2 = 0$ the spread is the rational expression

$$s(l_1, l_2) = \frac{(a_1b_2 - a_2b_1)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}.$$

The spread s is always a number between 0 and 1. It is 0 when the lines are parallel and 1 when the lines are perpendicular. An angle of $45°$ or $135°$ is a spread of $1/2$, and $30°$ or $150°$ and $60°$ or $120°$ are respectively spreads of $1/4$ and $3/4$. Figure 5 shows a spread protractor that you can download from the internet [1].

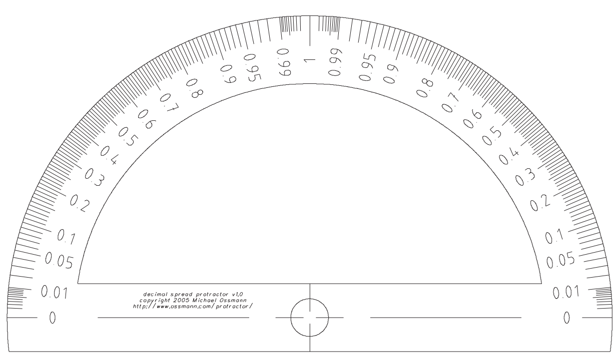

Figure 5: Spread protractor

Of course you need adjust to the fact that spread is not 'linear', but the advantages quickly become apparent when we look at the main laws of trigonometry expressed with the concepts of quadrance and spread instead of distance and angle. Transcendental functions are *not needed* to understand triangles!

Rational Trigonometry

The next two theorems are the rational analogs of the Sine law and the Cosine law. Both are proved completely independently from classical trigonometry.

Theorem 0.4 (Spread law) *Suppose the triangle* $\overline{A_1A_2A_3}$ *has quadrances* $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ *and* $Q_3 \equiv Q(A_1, A_2)$ *and spreads* $s_1 \equiv s(A_1A_2, A_1A_3)$, $s_2 \equiv s(A_2A_1, A_2A_3)$ *and* $s_3 \equiv s(A_3A_1, A_3A_2)$. *Then*

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}.$$

Theorem 0.5 (Cross law) *Suppose the triangle* $\overline{A_1A_2A_3}$ *has quadrances* $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ *and* $Q_3 \equiv Q(A_1, A_2)$ *and spreads* $s_1 \equiv s(A_1A_2, A_1A_3)$, $s_2 \equiv s(A_2A_1, A_2A_3)$ *and* $s_3 \equiv s(A_3A_1, A_3A_2)$. *Then*

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - s_3).$$

The reason for the terminology is that the quantity $1 - s_3 = c_3$ is called the **cross** between the two lines.

To prove these theorems, refer to either of the two diagrams in Figure 6.

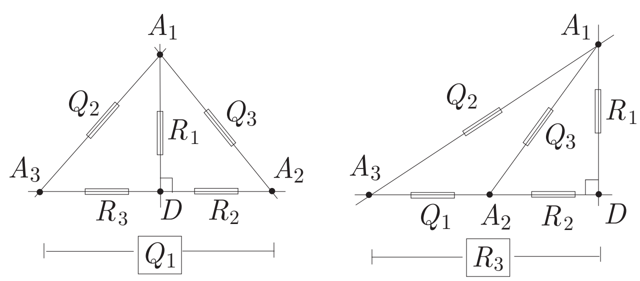

Figure 6: Spread law

The definition of the spreads at A_2 and A_3 gives

$$s_2 = R_1/Q_3 \qquad s_3 = R_1/Q_2. \tag{2}$$

Solve for R_1 to get

$$R_1 = Q_3s_2 = Q_2s_3$$

so that

$$\frac{s_2}{Q_2} = \frac{s_3}{Q_3}.$$

Symmetry then implies the Spread law.

Using (2) and Pythagoras' theorem, we have

$$\begin{aligned} R_1 &= Q_2 s_3 \\ R_3 &= Q_2 - R_1 = Q_2 (1 - s_3) \\ R_2 &= Q_3 - R_1 = Q_3 - Q_2 s_3 \end{aligned}$$

Since A_2, A_3 and D are collinear, apply the Triple quad formula to the three quadrances Q_1, R_2 and R_3, yielding

$$(Q_1 + R_3 - R_2)^2 = 4Q_1 R_3.$$

Substitute the values of R_3 and R_2, to get

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1 Q_2 (1 - s_3).$$

This establishes the Cross law.

There is one more main theorem of rational trigonometry—the analog of the fact that the sum of the angles of a triangle is approximately $3.141\,592\,65\ldots \approx \pi$.

Theorem 0.6 (Triple spread formula) *Suppose that a triangle $\overline{A_1 A_2 A_3}$ has spreads s_1, s_2 and s_3. Then*

$$(s_1 + s_2 + s_3)^2 = 2\left(s_1^2 + s_2^2 + s_3^2\right) + 4s_1 s_2 s_3.$$

To prove this, we combine the Spread law and the Cross law. From the Spread law, there is a non-zero number D, such that if Q_1, Q_2 and Q_3 are the corresponding quadrances of the triangle,

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3} \equiv \frac{1}{D}. \tag{3}$$

Rewrite the Cross law

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1 Q_2 (1 - s_3)$$

in the more symmetrical form

$$(Q_1 + Q_2 + Q_3)^2 = 2\left(Q_1^2 + Q_2^2 + Q_3^2\right) + 4Q_1 Q_2 s_3. \tag{4}$$

Use (3) to replace Q_1 by $s_1 D$, Q_2 by $s_2 D$ and Q_3 by $s_3 D$ in (4), and then divide by D^2. The result is

$$(s_1 + s_2 + s_3)^2 = 2\left(s_1^2 + s_2^2 + s_3^2\right) + 4s_1 s_2 s_3.$$

Example 0.1 *You might like to verify the main laws for the triangle with vertices $A_1 = [4,1]$, $A_2 = [1,2]$ and $A_3 = [2,4]$, with quadrances $Q_1 = 5$, $Q_2 = 13$ and $Q_3 = 10$. The corresponding spreads are $s_1 = 49/130$, $s_2 = 49/50$ and $s_3 = 49/65$. Several other interesting facts about this triangle are derived in [3].*

Remarkably, the five main laws—*Pythagoras' theorem*, the *Triple quad formula*, the *Spread law*, the *Cross law* and the *Triple spread formula*—suffice to solve the vast majority of trigonometric problems, invariably in a more accurate and simpler form that the classical theory which utilizes transcendental circular functions, as demonstrated at some length in [2]. Tables and calculators are not generally needed. Students can learn the new theory in a fraction of the time spent on the old one, and literally dozens of arcane facts and formulas can be relegated to the dusty shelves where they rightfully belong. The five main laws hold even in the geometry of special relativity. (I will explain this in a future article.)

Rational trigonometry cleanly separates the geometry of triangles from the mechanics of uniform motion around a circle. For the latter, transcendental circular functions such as $\sin\theta$ and $\cos\theta$ and their inverse functions are necessary, for the former they only complicate matters unnecessarily. The basic message: *square pegs don't fit into round holes*.

References

[1] M. Ossmann, 'Print a Protractor', available online at http://www.ossmann.com/protractor/

[2] N. J. Wildberger, *Divine Proportions: Rational Trigonometry to Universal Geometry*, Wild Egg Books, Sydney, 2005, http://wildegg.com.

[3] N. J. Wildberger, 'Survivor: The Trigonometry Challenge', posted 2006 at http://wildegg.com/authors.htm