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History of Mathematics: Ptolemy's Theorem

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This column is prompted by some correspondence with K R S Sastry, who for many years has been active in Mathematics, particularly Geometry. He has worked in his native India and also in Ethiopia, and has contributed prolifically to Mathematics journals over many years. In his query to me, he raised the question of the origins of a geometrical result known as Ptolemy's Theorem. The theorem concerns the situation depicted in Figure 1.

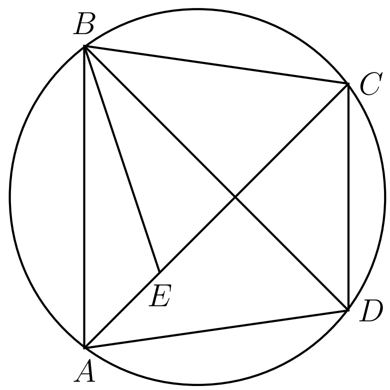

Figure 1

A, B, C and D are points lying on a circle and so arranged that the order of points as we traverse the circle (either clockwise or anticlockwise) is as given. (That is to say: AB and CD do not intersect inside the circle.) The theorem then states that

$$|AC|\,|BD| = |AB|\,|CD| + |BC|\,|DA|.$$

(Here I use $|AB|$ to mean the length of AB, etc.)

The theorem is a relatively elementary one, and I will provide a proof a little later on. Most such elementary geometric theorems are to be found in the collection that we know as Euclid's *Elements*.

Sastry's question to me was this: Was Ptolemy the originator of this theorem, or did he follow some earlier discoverer? This is a very reasonable thing to ask, since in a great many cases, the mathematicians whose names become attached to results are not in fact the discoverers.

But let us begin with some background.

Euclid lived in Alexandria (now in Egypt, but then part of the Greek empire) and was the leading mathematician of his day (a period some years either side of 300 BCE).

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His best-known work is the *Elements*, a massive collection (in 13 separate books) of the geometric knowledge of the time. Until very recently this work formed the basis of all school syllabuses of Geometry. It remains a towering achievement. The style of exposition and the insistence on formal demonstration of claimed results has done much to define the mathematical agenda for all subsequent time.

Ptolemy lived much later, in years either side of 100CE, and was a figure of comparable intellectual stature. He is not generally considered a mathematician in the same sense that Euclid was; rather he is classified as an astronomer and geographer. His major work was a compilation of the astronomical knowledge of his day. He published this as the *Syntaxis* (meaning 'compilation'), but his contemporaries and successors accorded it the title *megiste syntaxis* ('the greatest compilation') to distinguish it from other works they considered inferior.

Ptolemy also lived in Alexandria, and when the Arabs later occupied this city, they became heirs to much of its intellectual tradition. Many of the best works of the Greek mathematicians were translated into Arabic. (In many cases, this is the only form in which they survive.) Ptolemy's work was translated as *Al magest* ('the greatest') and this term was latter adopted by the Romans under the Latin name *Almagestum*. In English, we refer to it still as Ptolemy's *Almagest*.

There have been several English translations of this work, but the most recent and also the best is one by the very great historian of Mathematics, G J Toomer. His fine annotated translation first appeared in 1984. This makes it clear that Ptolemy did state and prove the theorem. In Toomer's translation it is to be found on p 50, but the convention has arisen in the study of Ptolemy's work of giving the page references from an earlier edition (by Heiberg). So the standard reference for Ptolemy's Theorem is H36.

Here is Ptolemy's proof. (Refer to Figure 1.)

Connect the point B to the line AC in such a way that the join BE makes the angle ABE equal to the angle CBD. Then to each of these angles add the angle EBD. This makes the angle ABD equal to the angle EBC. Now compare the triangle ABD with the triangle EBC. The angle ABD in the first of these is equal to the angle EBC in the second (just proved) and also the angle BDA in the first equals the angle BCE in the second (as a result of a result proved in Euclid's *Elements* (Book III, Proposition 21)). It follows that the angles BAD and BEC must also be equal (because the angles in a triangle must add to $180°$). Thus these two triangles are similar to one another; that is to say, they have the same shape so that one must be a scale-model of the other. This tells us that

$$\frac{|BC|}{|CE|} = \frac{|BD|}{|DA|}$$

and so $|BC|\,|DA| = |BD|\,|CD|$.

A similar argument can now be used to show that $|AC|\,|BD| = |BD|\,|AE|$. The result follows because $|AE| + |CE| = |AC|$.

There is a slight 'gap' in this proof. It was pointed out by Sir Thomas Heath, who included a discussion of Ptolemy's Theorem among the notes to his definitive edition of Euclid's *Elements*. This is illustrated by Figure 2, which shows that the point E may lie on the same side as C of the intersection of the two diagonals. In this case, instead of *adding* the angle EBD to each of the angles ABD and CBD, we subtract. After this, all goes well.

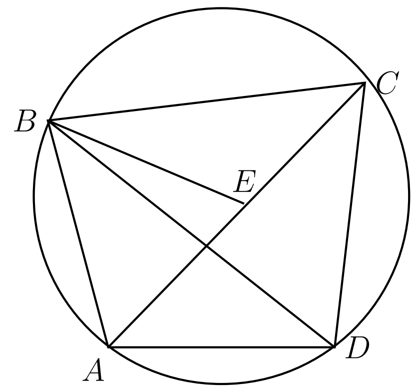

Figure 2

So, apart from a minor glitch, Ptolemy certainly stated and proved the theorem that now bears his name. But was he preceded by others? Was the result already known by the time Ptolemy wrote?

To look into this question, we first need to know the background to Ptolemy's interest in the configuration of Figure 1. The geometry of the circle is vitally important to the study of Astronomy and of Geography (the earth is round, and the heavens appear so). Nowadays, we use Trigonometry to discuss many of the quantitative aspects of Astronomy and Geodesy (the measurement of the earth), but Trigonometry had not yet been formulated.

Consider the arc and the chord AB in Figure 1 for example. Adopt the standard convention that the radius of the circle is 1. Then the arclength AB is simply the radian measure of the angle subtended by this arc at the centre of the circle (not drawn here). Call this angle θ. Then the length of the chord AB is given by $|AB| = 2\sin\frac{\theta}{2}$. However, back in Ptolemy's time, the functions $\sin$ and $\cos$ had not yet been developed. What took their place were a pair of other functions of θ, now no longer used. The first of these is the chord of θ, which we can write $ch\theta$, where

$$ch\theta = |AB| = 2\sin\frac{\theta}{2}.$$

(The other function used is now called the versed sine of θ, written $vs\theta$, where in today's notation, $vs\theta = 1 - \cos\theta$. If you experiment with the use of the functions $ch\theta$ and $vs\theta$ in place of our familiar trig functions, you will soon see the benefits of our modern approach!)

In place of trig tables (now replaced by calculators or computers) to supply the values of the trig functions, the ancients needed 'chord tables'. This is what Ptolemy was busily involved with. His theorem, applied to various special cases enables the construction of such tables with a reduced amount of computational time and labor.

So, if we try to find earlier discoveries of Ptolemy's Theorem, the sensible place to look for them is among the chord tables produced before Ptolemy's time. There are two possibilities. Earlier chord tables were produced by the Greek astronomers Hipparchus and Menelaus.

Hipparchus (of Rhodes) lived in the second century BCE, and was a major figure in the history of Astronomy. Very little is known of his life. There was only one (minor) work from his hand, and what little we know about his work derives from Ptolemy, who was concerned to build upon it. Toomer has this to say:

> ... although Ptolemy obviously had studied Hipparchus's writings thoroughly and had a deep respect for his work, his main concern was not to transmit it to posterity but to use it and, where possible, improve upon it in constructing his own astronomical system.

Menelaus lived in the years either side of 100CE and was the author of many books, most of which were on Geometry. Nearly all have been lost, only Sphaerica (it) survives in something like its entirety. This is a book on the geometry of the sphere, with applications to Astronomy. (His name is now attached to a theorem about ordinary – plane – Geometry, but that is another story!)

Both Hipparchus and Menelaus produced chord-tables, so it is possible that their (poorly preserved) texts included accounts of Ptolemy's Theorem. However, Toomer argues against this idea. In a note on p 50 of his translation, he suggests that these authors used other simpler means to the same ends. His reasons are highly technical and take us much further afield. For this reason, I will not go into them here, but interested readers may follow the leads given in his note.

There are some interesting theorems allied to Ptolemy's. The most straightforward concerns the case in which the points A, B, C and D do not all lie on a circle. Look at Figure 3.

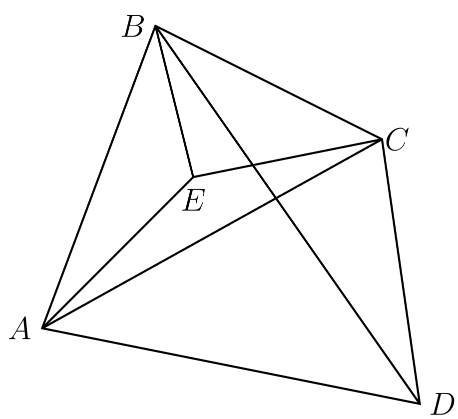

Figure 3

No circle is drawn, because the points in question do not all lie on one. However, we proceed much as we did in the earlier case. At A make an angle BAE equal to angle BDC and at B make an angle ABE equal to angle CBD. E is the point where the lines defining these angles meet. Now join EC.

The principal difference between this case and the last is that the point E now will not lie on the line AC. We can still argue that

$$|BC|\,|DA| = |BD|\,|CE| \quad \text{and that} \quad |AC|\,|BD| = |BD|\,|AE|$$

although the reasoning is a little different. However, when we come to the final step, we still find that

$$|BC|\,|DA| + |AC|\,|BD| = |BD|\,|CE| + |BD|\,|AE|.$$

But now because this time $|AE| + |CE| > |AC|$, we have

$$|AB|\,|CD| + |BC|\,|DA| > |AC|\,|BD|.$$

In my schooldays, we used a different proof of Ptolemy's Theorem. This made use of the Cosine Rule. As now stated, this involves the trigonometric ratios, and so would not qualify as an early candidate. However, results exactly equivalent to the Cosine Rule (but using different language) in fact occur in Euclid. Two separate cases occur as Propositions 12 and 13 of Book II of the *Elements*. For the convenience of modern readers, I will indulge in a minor anachronism and present the proof in modern dress.

To make the work simpler to follow, introduce the notation

$$|AB| = a,\ |BC| = b,\ |CD| = c,\ |DA| = d,\ |AC| = x,\ |BD| = y.$$

Write A for the angle DAB, and so on for the other angles of the original quadrilateral.

Then

$$x^2 = a^2 + b^2 - 2ab\cos B,$$

by the Cosine Rule; but also

$$x^2 = c^2 + d^2 - 2cd\cos D = c^2 + d^2 + 2cd\cos B,$$

because of the property of cyclic quadrilaterals (those inscribed in circles) that tells us that their opposite angles add to $180°$ (Proposition 22 of Book III of the *Elements*). A bit of algebra follows. I leave it to readers, but the result is that, when B is eliminated, then

$$x^2 = \frac{(ac+bd)(ad+bc)}{ab+cd}.$$

Similarly $\quad y^2 = \dfrac{(ac+bd)(ab+cd)}{ad+bc}.$

Now we may multiply and take a square root to find $xy = ac + bd$, which is Ptolemy's Theorem.

When Mr Sastry wrote, he wondered whether the strong Indian interest in the properties of the quadrilateral might also have contributed to the story. The principal name here is that of Brahmagupta, who lived in the seventh century CE, and thus came a lot later than the other figures we have been discussing. His name is attached to a formula for the area of a cyclic quadrilateral. Write $s = \dfrac{1}{2}(a+b+c+d)$ (s stands for 'semi-perimeter'), and A for area. Then

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

It was once widely stated that Brahmagupta erroneously held that this formula applied to all quadrilaterals. Indeed I still possess a school textbook (Durell & Robson's *Advanced Trigonometry*) saying just this. However, more recent research leads us to the view that he was in fact well aware of the restriction.

If we allow the length of one side of the quadrilateral (d say) to shrink to zero, then the result is a triangle, and all triangles are cyclic, so the result applies to all triangles. We in the West know this as Heron's formula. Heron lived in the first century CE and in his book *Metrica* included the formula which we now write as

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

for the area of a triangle with side-lengths a, b, c.

It would be possible, but extremely tedious, to prove Ptolemy's Theorem by appeal to the Heron and Brahmagupta formulae. If you try to go down this path, you will probably soon see the inefficiency of the process.

However, in other areas the Indian mathematicians do seem to have been the first to find the relevant results. Earlier, to expedite a point in the discussion, I chose to set the radius in which the quadrilateral was inscribed as 1. In general, however, this cannot be done without restricting the sizes of the sides in some way. If we prefer to allow general sides, subject to the cyclic restriction only, then we need a formula for the radius, R, of the circle containing the quadrilateral.

Given a, b, c and d, and allowing the angles to be suitably chosen can always result in a cyclic configuration, which in fact maximizes the area of the quadrilateral. This is a special case of a result known as Fasbender's Theorem. I wrote on it in *Function* in June 1996.

The formula for R is:

$$R = \sqrt{\frac{(ab+cd)(ac+bd)(ad+cd)}{(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d)}}$$

It was usual to attribute this result to the Swiss mathematician Simon L'Huillier (1750-1840), but it is now recognized that it was stated much earlier by the Indian mathematician Parameśvara (around 1430 CE). This was pointed out by Radha Charan Gupta in a 1977 article in the specialist journal *Historia Mathematica.* (I am indebted to Mr Sastry for a copy.)

I will not give a proof here of Parameśvara's formula. My old school text has one, but Gupta's article presents a different one, one which uses a result very like Ptolemy's Theorem along the way. It will be recognized that the denominator of Parameśvara's formula for R contains the elements of Brahmagupta's area formula. The use of this formula is a crucial part of both the proofs I have seen.

There is much else that can be said about quadrilaterals. To explore the matter further, look at the article on 'quadrilateral' at the Mathworld website

http://mathworld.wolfram.com/Quadrilateral.html

and the references given there. Readers may be surprised that so many complicated theorems can be proved about so simple a figure as a quadrilateral, but then they might consider that an even simpler figure is the triangle. And whole books have been written about that!