Parabola Volume 43, Issue 3 (2007)

Solutions to Problems 1241–1250

Q1241 Show that Simpson's Elementary Rule

$$\int_a^b f(x)dx \approx \left(\frac{b-a}{6}\right)\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

is an exact equality for the quadratic function

$$f(x) = Ax^2 + Bx + C.$$

ANS: (correct answers submitted by Julius Guest, Victoria)

$$\int_a^b \big(Ax^2+Bx+C\big)\,dx$$

$$\begin{aligned}
&= \left(\frac{Ax^3}{3}+\frac{Bx^2}{2}+Cx\right|_a^b \\
&= \frac{A(b^3-a^3)}{3}+\frac{B(b^2-a^2)}{2}+C(b-a) \\
&= \frac{(b-a)}{6}\big(2A(a^2+ab+b^2)+3B(b+a)+6C\big) \\
&= \left(\frac{b-a}{6}\right)\big(Aa^2+Ba+C \\
&\qquad +Ab^2+Bb+C \\
&\qquad +Aa^2+Ab^2+4C+2Aab+2Bb+2Ba\big) \\
&= \left(\frac{b-a}{6}\right)\left(Aa^2+Ba+C+Ab^2+Bb+C\right. \\
&\qquad \left.+4\left(A\frac{(a^2+b^2+2ab)}{4}+B\left(\frac{a+b}{2}\right)+C\right)\right) \\
&= \left(\frac{b-a}{6}\right)\left(Aa^2+Ba+C+Ab^2+Bb+C\right. \\
&\qquad \left.+4\left(A\left(\frac{a+b}{2}\right)^2+B\left(\frac{a+b}{2}\right)+C\right)\right) \\
&= \left(\frac{b-a}{6}\right)\left(f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right).
\end{aligned}$$

Q1242 Use the laws of addition and multiplication of complex numbers (see the first article in this issue) to find the complex numbers z that satisfy

$$z^3 = i.$$

ANS: Let $z = a + ib$ where a and b are real numbers then

$$\begin{aligned} z^3 = i \quad &\Rightarrow \quad (a+ib)^3 = i \\ &\Rightarrow \quad (a+ib)(a+ib)^2 = i \\ &\Rightarrow \quad (a+ib)(a^2 - b^2 + 2aib) = i \\ &\Rightarrow \quad a^3 - ab^2 + 2a^2ib + ia^2b - ib^3 - 2ab^2 = i \\ &\Rightarrow \quad (a^3 - 3ab^2) + (3a^2b - b^3)i = i \end{aligned}$$

This results in two simultaneous equations to solve for a and b:

$$a^3 - 3ab^2 = 0 \qquad (1)$$

$$3a^2b - b^3 = 1. \qquad (2)$$

If we consider $a = 0$ in Equation (1) then $b = -1$ in Equation (2) and $z = -i$ is a solution. If however $a \neq 0$ then we require $a^2 = 3b^2$ or $a = \pm\sqrt{3}b$ in Equation (1) and then $b = \frac{1}{2}$ in Equation (2) so that $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $z = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ are also solutions.

Q1243 Prove that if

$$x^n + y^n = z^n$$

where x, y, z, n are all positive integers and $x \leq y < z$ then it follows that $y > n$.

ANS: We will prove this by using the method of contradiction.

First we assume $y \leq n$.

Now we are given

$$z > y \quad \text{so that} \quad z \geq y + 1$$

and hence

$$z^n \geq (y+1)^n > y^n + ny^{n-1} \quad \text{(from the binomial theorem).}$$

From our assumption $n \geq y$ we have $ny^{n-1} \geq y^n$ so that we now have

$$z^n > 2y^n.$$

But $y \geq x$ so it must also be the case that

$$z^n > 2x^n.$$

If we now add the two inequalities for z^n we have

$$2z^n > 2y^n + 2x^n$$

and hence $z^n > y^n + z^n$ which is in contradiction with the requirement $x^n + y^n = z^n$. Thus we must conclude that $y \not\leq n$, i.e. $y > n$ as required.

Q1244 In modular arithmetic if a, b and c are integers then

$$a \equiv b \pmod{c}$$

if $(a-b)$ is an integer multiple of c. Find the integer $n > 0$ that satisfies

$$n(n-1) \equiv n-2 \pmod{n+1}.$$

ANS: (correct answer submitted by John Colin Barton, Victoria)

If $n(n-1) \equiv (n-2)(\text{mod } n+1)$ then $n(n-1) = p(n+1)+n-2$ where p is an integer.

We thus require

$$n^2 - 2n + 2 = p(n+1).$$

But we can factor

$$n^2 - 2n + 2 = (n-3)(n+1) + 5$$

so that we require

$$5 = q(n+1)$$

where q is also an integer. It follows that $n = 4$; and $4(3) = 2 \pmod{5}$.

Q1245 Estimate the area of a semi-circular disc of radius r using Simpson's Elementary Rule

$$\int_a^b f(x)dx = \left(\frac{b-1}{6}\right)\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).$$

ANS: The Cartesian equation for a circle of radius r is $x^2 + y^2 = r^2$ from which we define $f(x) = y = \sqrt{r^2 - x^2}$ with $a = -r$ and $b = +r$.

The area of the semi-circular disc is thus

$$\begin{aligned} A = \int_{-r}^{+r} \sqrt{x^2 - r^2}dx &= \left(\frac{2r}{6}\right)(f(-r) + 4f(0) + f(r)) \\ &= \frac{r}{3}(0 + 4r + 0) \\ &= \frac{4r^2}{3}. \end{aligned}$$

Compare with the exact result $A = \frac{\pi}{2}r^2$.

Q1246 A farmer has a dog, a chicken and a bucket of grain to take across a river in a boat. The farmer can only take one item at a time in the boat. The dog cannot be left alone with the chicken as it would eat it. The chicken cannot be left alone with the grain as it would eat it.

Find two ways in which the farmer can transport the dog, chicken and grain safely across the river.

ANS: The farmer takes the chicken across and then returns alone.

The farmer takes the dog across and leaves the dog but returns with the chicken.

The farmer takes the grain across and returns alone. The farmer takes the chicken across.

The second solution is obtained by swapping the order of taking the dog across and the grain across.

Q1247 A game show contestant has a chance to win a prize by guessing which of five boxes the prize is contained in.

After the contestant announces their guess the game show host points to one of the remaining boxes and informs the contestant that the prize is not in this box. The host then asks the contestant if they would like to stick with their original guess or change their selection.

What is the best strategy for the contestant and what is the probability of winning the prize with this strategy?

ANS: There is a probability $\frac{1}{5}$ that the initial guess is correct and a probability $\frac{4}{5}$ that the prize is in one of the remaining boxes. After the host indicates one of the four remaining boxes as not containing the prize the three remaining boxes must now have the prize with probability $\frac{4}{5}$. Thus the contestant should take up the host's offer to change their selection boosting their chance of winning from $\frac{1}{5} = \frac{3}{15}$ to $\frac{1}{3} \times \frac{4}{5} = \frac{4}{15}$.

Q1248 Show that if $f(x)$ is a continuous solution of the functional equation

$$2f(x)f(y) = f(x+y) + f(x-y)$$

then $f(x)$ is an even function, i.e. $f(x) = f(-x)$.

ANS:(correct answer submitted by John Colin Barton, Victoria)

First note that if $x = 0$ and $y = 0$ then

$$2f(0)f(0) = f(0) + f(0)$$

$$\text{i.e.} \qquad (f(0))^2 = f(0)$$

so that either $f(0) = 0$ or $f(0) = 1$. If $f(0) = 0$ is a solution then for any x we have

$$0 = 2f(x)f(0) = f(x+0) + f(x-0) = 2f(x)$$

so that $f(x) = 0$ for any x.

This is a trivial solution so let us now suppose $f(x) \neq 0$ for all x but $f(0) = 1$. We now have that for any y

$$2f(0)f(y) = f(0+y) + f(0-y)$$

so that

$$2f(y) = f(y) + f(-y)$$

and $f(y) = f(-y)$ for any y so that $f(x)$ is an even function as required.

Q1249 One of the most famous numbers is the golden number ϕ which can be written as

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}.$$

A less famous number is the silver number (or plastic number) p which can be written as

$$p = \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \cdots}}}\,.$$

Show that

$$\phi = \frac{1 + \sqrt{5}}{2}$$

and

$$p = \sqrt[3]{\sqrt{\frac{23}{108}} + \frac{1}{2}} - \sqrt[3]{\sqrt{\frac{23}{108}} - \frac{1}{2}}\,.$$

ANS: (correct answer submitted by Julius Guest, Victoria and David Shaw, Geelong)

First note that if $\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}$, then $\phi = \sqrt{1 + \phi}$ so that $\phi^2 = 1 + \phi$ and $\phi^2 - \phi - 1 = 0$. We now solve this quadratic for the positive solution to obtain

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

In a similar fashion, given that $p = \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \cdots}}}$ we can write

$$p = \sqrt[3]{1 + p}$$

so that

$$p^3 - p - 1 = 0.$$

This cubic equation can be solved using the method of Cardan (or Tartaglia). The solution to the cubic $x^3 + ax - b = 0$ is:

$$\begin{aligned} x &= \sqrt[3]{\sqrt{\left(\frac{a}{3}\right)^2 + \left(\frac{b}{2}\right)^2} + \frac{b}{2}} \\ &- \sqrt[3]{\sqrt{\left(\frac{a}{3}\right)^2 + \left(\frac{b}{2}\right)^2} - \frac{b}{2}} \end{aligned}$$

Now substitute $a = -1$ and $b = 1$ to write

$$p = \sqrt[3]{\sqrt{\frac{23}{108}} + \frac{1}{2}} - \sqrt[3]{\sqrt{\frac{23}{108}} - \frac{1}{2}}\,.$$

Q1250 Find the positive integer x that satisfies the identity

$$x = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}}.$$

ANS: It is useful to define

$$x(k) = \sqrt{1 + k\sqrt{1 + (k+1)\sqrt{1 + (k+2)\sqrt{1 + \cdots}}}}$$

then we have

$$x^2(k) = 1 + kx(k+1)$$

and

$$x(0) = 1.$$

Note that it follows that $x(k) = k + 1$ if $k = 0$. We could try this as a general solution. If we substitute $x(k) = k + 1$ into $x^2(k) = 1 + kx(k+1)$ we find

$$(k+1)^2 = 1 + kx(k+1)$$

so that $x(k+1) = k + 2$ and it follows by induction that $x(k) = k + 1$. The special case $k = 2$ yields

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}} = 3.$$

The following answer has been provided by John Colin Barton from Victoria:

Let m denote a positive integer greater than one. Then

$$\begin{aligned}
m &= \sqrt{m^2} \\
&= \sqrt{1 + (m^2 - 1^2)} \\
&= \sqrt{1 + (m-1)\sqrt{(m+1)^2}} \\
&= \sqrt{1 + (m-1)\sqrt{1 + (m+1)^2 - 1^2}} \\
&= \sqrt{1 + (m-1)\sqrt{1 + m\sqrt{(m+2)^2}}} \\
&= \sqrt{1 + (m-1)\sqrt{1 + m\sqrt{1 + (m+2)^2 - 1^2}}} \\
&= \sqrt{1 + (m-1)\sqrt{1 + m\sqrt{1 + (m+1)\sqrt{1 + (m+3)^2 - 1^2}}}} \\
&= \sqrt{1 + (m-1)\sqrt{1 + m\sqrt{1 + (m+1)\sqrt{1 + (m+2)\sqrt{1 + \ldots}}}}}
\end{aligned}$$

where the ... indicates a continuation of the pattern. The special case of $m = 3$ yields the equation in Q1250 with $x = 3$.

John has also correctly pointed out that the question would have been better worded as, "Show that there is a positive integer x expressible as

$$x = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}}.$$

Find x."