Parabola Volume 44, Issue 1 (2008)

Mathematical Induction

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The method known as *mathematical induction* is generally thought to have been introduced by Pascal (circa 1654), although a contrapositive form called the 'method of infinite descent' was used by Fermat a little earlier. The name 'mathematical induction' was first used by De Morgan.

Mathematical Induction is a method used to prove an **infinite number** of propositions.

We can write these propositions using the notation $P(n)$, where n is a positive integer.

For example, to prove that $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$, we can write:

For each positive integer n let $P(n)$ denote the proposition $1+2+\cdots+n = \frac{1}{2}n(n+1)$.

While this may take a little getting used to, it makes the induction much easier to set out and avoids the use of such meaningless statements as *it is true for* $n = 1$. (I am never quite sure what *it* is.) I am not overly pedantic on this point, but if you are able to cope with this notation, it is very convenient, and clearly emphasises exactly what we are doing.

The simplest form of the *principle of induction* is:

(i) Show that $P(1)$ is true

(ii) Show that the truth of $P(k)$ implies the truth of $P(k + 1)$.

The induction principle states that we can then conclude that $P(n)$ is true for all integers greater than or equal to 1.

(An alternative way to view this, is via set theory. Define a set $S = \{n \in \mathbf{N} : P(n)$ is true $\}$. We then show that $1 \in S$ and that if $k \in S$ then $k + 1 \in S$ and the principle of induction then allows us to claim that $S = \mathbf{N}$ (assuming $0 \notin \mathbf{N}$).)

It is firstly easier to think of the standard analogy of a string of dominoes which are arranged in such a way that if any given domino is knocked over then it in turn knocks over the next one. This analogy is a good one but it is only an analogy, and we have to remember that in the domino situation there is only a **finite number** of dominoes.

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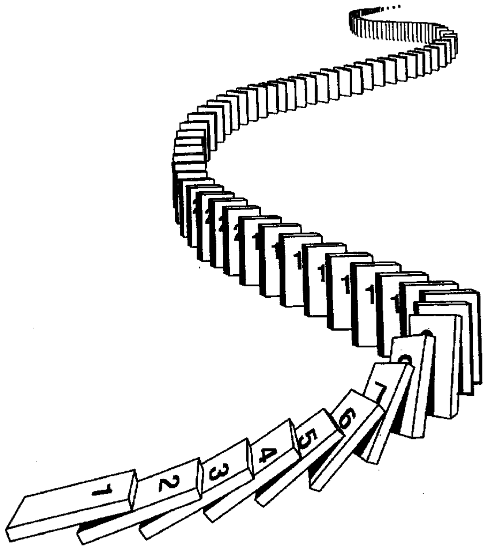

The first thing to say then about induction, is that you should think of the principle of induction as 'God-given' or if you are an atheist or agnostic, you might prefer to think of it as an *axiom.* In other words, the principle of induction cannot be proven (although it's reasonableness can be justified) without assuming some equivalent form of it, usually the Well-Ordering Principle. This is important to realise since some school books now insist that inductions end with the following mantra:

The statement is true for 1 so true for 2, true for 2 so true for 3,... *hence* true for all integers $n \geq 1$.

The use of *hence* here is very misleading, since it is not the preceding words that give the result, but the *principle of induction.* The last step in an inductive proof should be:

Hence the statement (or $P(n)$) is true for all $n \geq 1$ by induction.

An inductive proof may be set out as follows:

1. Carefully define the family of propositions which are to be proven.

2. Prove that $P(1)$ is true.

3. Assuming the truth of $P(k)$ for some particular integer k, $k \geq 1$, we deduce the truth of $P(k+1)$.

4. Conclude the truth of $P(n)$ for all $n \geq 1$ by induction.

Some Examples:

1. The standard first example which is often done is to prove that for all integers $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

Let $P(n)$ be the above proposition.

$P(1)$ is clearly true since $1^2 = \frac{1}{6} \times 1 \times 2 \times 3$.

Let k be an integer for which $P(k)$ is true, that is:

$$1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1).$$

Then $1^2+2^2+\cdots+k^2+(k+1)^2 = \frac{1}{6}k(k+1)(2k+1)+(k+1)^2$ and some basic algebra leads to $\frac{1}{6}(k+1)(k+2)(2k+3)$ and so the proposition $P(k+1)$ is true. Hence the result is true for all $n \geq 1$ by induction.

2. Here is an example from the 2001 HSC paper:

Prove that for all integers n, $n \geq 1$, we have $n^3+(n+1)^3+(n+2)^3$ is divisible by 9.

Let $P(n)$ be the given proposition. It is easy to show that $P(1)$ is true since 36 is divisible by 9.

Let k be an integer for which $P(k)$ is true. That is, we suppose that

$$k^3+(k+1)^3+(k+2)^3 = 9L$$

for some integer L. Then

$$(k+1)^3+(k+2)^3+(k+3)^3 = 9L-k^3+(k+3)^3 = 9L+9k^2+27k+27$$

which is clearly divisible by 9. Hence $P(k+1)$ is true and so $P(n)$ is true for all integers $n \geq 1$ by induction.

3. Not all inductions necessarily begin at the case of $n=1$ or $n=0$.

Prove that $n! > 3^n$ for $n \geq 7$.

(Note that the inequality is false until $n=7$). The induction then, will start from the case $n=7$.

To complete the inductive proof, we assume the result true for some integer (≥ 7) and write

$$(k+1)! = (k+1)k! > (k+1)3^k > 3 \times 3^k = 3^{k+1}$$

and the induction is complete.

4. Here is a slightly more difficult induction problem.

Prove that $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{3^n} > \frac{5}{6}(n+1)$ for all integers $n \geq 1$.

Let $P(n)$ be the given proposition which is clearly true for $n=1$.

Suppose $P(k)$ is true, then

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{3^k} > \frac{5}{6}(k+1).$$

Now consider $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{3^{k+1}}$. Notice that this sum has an extra 2×3^k terms. We break up the sum as follows:

$$\begin{aligned}1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{3^k}& \\ &+\underbrace{\frac{1}{3^k+1}+\frac{1}{3^k+2}+\cdots+\frac{1}{2\times 3^k}} \\ &\qquad +\underbrace{\frac{1}{2\times 3^k+1}+\frac{1}{2\times 3^k+2}+\cdots+\frac{1}{3^{k+1}}}\end{aligned}$$

Now each of the 3^k terms in the first brace are greater than the last term in the group and similarly with the second group, hence this sum is

$$> \frac{5}{6}(k+1)+3^k\times\frac{1}{2\times 3^k}+3^k\times\frac{1}{3^{k+1}} = \frac{5}{6}(k+1)+\frac{1}{2}+\frac{1}{3} = (k+2)\frac{5}{6}$$

and so $P(k+1)$ is true and hence $P(n)$ is true for all $n \geq 1$ by induction.

(Note that this result shows that the *harmonic series* $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges to infinity.)

5. In most induction problems the 'easy part' is to prove the initial case and the 'hard part' is the inductive step. This is not always so.

For example, prove that for any given integer $k > 2$, if $n \geq k^2$ then $k^n < n!$.

The inductive part is very easy, since if we assume the statement is true for some particular integer n, then $k^{n+1} = kk^n < kn!$ by assumption, but since $n \geq k^2$, we have $k < (n+1)$ and so $k^{n+1} < (n+1)n! = (n+1)!$

The difficult part is to get the induction started, since we have to show that the statement is true for $n = k^2$, i.e. we have to show that $k^{k^2} < (k^2)!$ which is rather difficult to prove.

Let us now look at one of the variations of the inductive method.

6. The following problem is due to Erdös (et al.) from about the 1960's.

Every positive integer n can be written in the form

$$\pm 1^2 \pm 2^2 \pm 3^2 \pm 4^2 \pm ... \pm m^2$$

for some integer m.

For example

$$\begin{aligned}
0 &= 1^2 + 2^2 - 3^2 + 4^2 - 5^2 - 6^2 + 7^2 \\
1 &= 1^2 \\
2 &= -1^2 - 2^2 - 3^2 + 4^2 \\
3 &= -1^2 + 2^2 \\
4 &= -1^2 - 2^2 + 3^2
\end{aligned}$$

and so on.

Now instead of going from the kth case to the $(k+1)$th case, we will go to the $(k+4)$th case. We assume the truth of the k case, i.e. we assume that k can be written as $\pm 1^2 \pm 2^2 \pm ... \pm p^2$ for some choice of the signs.

We can write 4 as

$$4 = (m+1)^2 - (m+2)^2 - (m+3)^2 + (m+4)^2$$

which is true for any m and so choosing $m = p$ we can write

$$k + 4 = \underbrace{\pm 1^2 \pm 2^2 ... \pm p^2} + \underbrace{(p+1)^2 - (p+2)^2 - (p+3)^2 + (p+4)^2}$$

and so we have written $k+4$ in the given form.

Now, we showed that the result was true for $n = 0, 1, 2, 3$ and so by the inductive step, we know it is true for the next four numbers, and thus by an extension of the inductive principle, the result is true for all n.

7. Induction does not always work!

For example, try to prove that

$$\frac{1}{2} \times \frac{3}{4} \times ...... \times \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}.$$

for all integers $n \geq 1$. The statement is certainly true for $n = 1$. If we let $P(n)$ be the given proposition and assume the statement $P(k)$ is true, then to show $P(k+1)$ is true we have to show that

$$\frac{1}{\sqrt{3k}} \times \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+3}}$$

but this is equivalent to $3k+3 < 0$ which is FALSE! The induction method doesn't appear to work here.

However, if we tighten the inequality to

$$\frac{1}{2} \times \frac{3}{4} \times .... \times \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$$

then the induction does work, since the desired inequality reduces to $19k < 20k$ which is true.

Strong Induction:

As you can see there are many variations of the inductive idea. As well as those mentioned above, there is also *strong induction.*

The simplest version of this, applied to a family of propositions $P(n)$ is:

(1) Show $P(1)$ **and** $P(2)$ are true.

(2) Assume **both** $P(k)$ and $P(k+1)$ are true and from this deduce the truth of $P(k+2)$.

The stronger *principle of induction* then asserts that $P(n)$ is true for all $n \geq 1$.

(Note: It can be shown that any proposition which is shown to be true by strong induction can also be shown to be true by ordinary induction, but the argument will be more difficult.)

Here is an example:

8. Suppose that a_n is a sequence of numbers satisfying $a_1 = 12, a_2 = 30$ and
$a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 3$, prove that $a_n = 3 \times 2^n + 2 \times 3^n$.

Let $P(n)$ be the above proposition, then $P(1)$ and $P(2)$ are clearly true so suppose that $P(k)$ and $P(k+1)$ are true. That is,

$$a_k = 3 \times 2^k + 2 \times 3^k, \qquad a_{k+1} = 3 \times 2^{k+1} + 2 \times 3^{k+1}.$$

Now consider a_{k+2}. By the definition of a_n we can write

$$a_{k+2} = 5a_{k+1} - 6a_k$$

and using the inductive assumption,

$$a_{k+2} = 5(3 \times 2^{k+1} + 2 \times 3^{k+1}) - 6(3 \times 2^k + 2 \times 3^k)$$

which re-arranges to give

$$3 \times 2^{k+2} + 2 \times 3^{k+2}$$

and so $P(k+2)$ is true. Hence $P(n)$ is true for all $n \geq 1$ by (strong) induction.

Let me finish with some harder problems:

9. In the 4-unit paper in 1998 students were asked to prove by induction the so-called AM-GM inequality, which says that the Arithmetic mean of a set of positive real numbers is greater than their geometric mean. It is worthwhile going through the proof of this:

Here is the question:

Suppose p, q, s are fixed and positive real numbers, with $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

a. By considering the function $f(t) = \frac{s^p}{p} + \frac{t^q}{q} - st$, show that for all $t > 0$ we have

$$\frac{s^p}{p} + \frac{t^q}{q} \geq st.$$

b. Use (a) and induction to prove that

$$(x_1 x_2 .... x_n)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + ... + x_n}{n},$$

for all $x_1, x_2, ..., x_n > 0$ and n an integer greater or equal to 1.

c. Deduce that for all real numbers $y_1, y_2, ..., y_n > 0$ we have

$$\frac{y_1}{y_2} + \frac{y_2}{y_3} + ... + \frac{y_{n-1}}{y_n} + \frac{y_n}{y_1} \geq n.$$

Here is a solution:

Note that the given relationship between p and q can also be written as $p = \frac{q}{q-1}$. Now $f'(t) = t^{q-1} - s = 0$ for a stationary point, so $t = s^{\frac{1}{q-1}}$.

$$f(s^{\frac{1}{q-1}}) = \frac{s^p}{p} + \frac{s^{\frac{q}{q-1}}}{q} - s.s^{\frac{1}{q-1}} \qquad = \frac{s^p}{p} + \frac{s^p}{q} - s^{1+\frac{1}{q-1}} = s^p - s^{\frac{q}{q-1}} = 0.$$

Also $f''(t) > 0$ for $t > 0$ so we have a minimum at $t = s^{\frac{1}{q-1}}$. Thus the graph of the curve looks a little bit like a 'quadratic' with a minimum turning point at $(s^{\frac{1}{q-1}}, 0)$, at least for $t \geq 0$. Hence for $t > 0$ we have $f(t) \geq 0$ and so the result follows.

ii) Let $P(n)$ be the given proposition for each integer n. $P(1)$ is obviously true so suppose $P(k)$ true for some particular integer k. Consider

$$(x_1 x_2 .... x_k x_{k+1})^{\frac{1}{k+1}} = \left[(x_1 x_2 ... x_k)^{\frac{1}{k}}\right]^{\frac{k}{k+1}} x_{k+1}^{\frac{1}{k+1}}$$

$$\leq \left(\frac{x_1 + x_2 + ... + x_k}{k}\right)^{\frac{k}{k+1}} x_{k+1}^{\frac{1}{k+1}}$$

by the inductive hypothesis. Now we use (i), with s, t being the two terms being multiplied above and $p = \frac{k+1}{k}, q = k+1$. Observe that $p = \frac{q}{q-1}$ as required and that $p, q > 1$ since $k \geq 1$.

We can thus write

$$LHS \leq \frac{\left(\frac{x_1 + x_2 + ... + x_k}{k}\right)}{\frac{k+1}{k}} + \frac{x_{k+1}}{k+1} = \frac{x_1 + x_2 + ... + x_{k+1}}{k+1}$$

whence we have $P(k+1)$ is true and the result follows by induction for all $n \geq 1$.

iii) Let $P = \frac{y_1}{y_2} + \frac{y_2}{y_3} + ... + \frac{y_n}{y_1}$ then using $x_1 = \frac{y_1}{y_2}, x_2 = \frac{y_2}{y_3} ..., x_n = \frac{y_n}{y_1}$ in the inequality in question (iii), we can write

$$P \geq n \left(\frac{y_1 y_2 ... y_n}{y_2 y_3 .... y_n y_1} \right)^{\frac{1}{n}} = n.$$

There are several other proofs of part (ii) above, but let me show you a rather subtle and shorter proof that also uses induction.

Suppose that $\{x_1, x_2, ..., x_n\}$ is any set of positive real numbers whose product is 1, that is $x_1 x_2 ..... x_n = 1$. Prove by induction that $x_1 + x_2 + ... + x_n \geq n$.

The case $n = 1$ is trivial, so suppose that the statement is true for some particular n and consider the case of $n+1$ numbers, $x_1, x_2, ... x_n, x_{n+1}$ whose product is 1. Clearly not all the numbers $x_1, ..., x_n, x_{n+1}$ can be greater than 1 and nor can they be all less than 1, so label them so that $x_n < 1$ and $x_{n+1} > 1$. Now $x_1 x_2 .....(x_n x_{n+1}) = 1$ and so the n numbers $x_1, x_2, ..., (x_n x_{n+1})$ have a sum greater than n by our inductive assumption, i.e.

$$n \leq x_1 + x_2 + ... + x_n x_{n+1}.$$

Now since $x_n < 1$ and $x_{n+1} > 1$, by writing $x_n = 1 - \epsilon$ and $x_{n+1} = 1 + \delta$, $(\epsilon, \delta > 0)$, we can show that $x_n + x_{n+1} > 1 + x_n x_{n+1}$. Substituting into the above we have

$$n \leq x_1 + ... + x_n + x_{n+1} - 1 \leq x_1 + ... + x_n + x_{n+1}$$

from which the $(n+1)$th case follows. Hence the result is true by induction.

If we replace each x_i with $\frac{a_i}{\sqrt[n]{a_1 a_2 ... a_n}}$ we have $x_1 x_2 .... x_n = 1$ and so

$$x_1 + x_2 + ... + x_n = \frac{a_1 + a_2 + ... + a_n}{\sqrt[n]{a_1 a_2 .. a_n}} \geq n$$

from which we have

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$$

which is the AM-GM inequality.

10. Some difficult induction problems can be solved by actually making the problem more general.

For example, it is notoriously difficult to prove by induction the following identity involving the Fibonacci numbers:

$$F_{2n+1} = F_{n+1}^2 + F_n^2$$

but it is much easier to prove $F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$ by induction on n (fixing m) and then specialising by putting $m = n$. To prove the more general result, we need strong induction and I leave you to check the details.