Parabola Volume 44, Issue 2 (2008)

UNSW School Mathematics Competition

Problems and Solutions
Junior Division

Problem 1

1. What is the angle between the long hand and the short hand of a clock at twenty minutes past four?
2. What is the next time, to the nearest second, at which the two hands of the clock have the same angle between them?

Solution

1. At twenty past four the short hand is $\frac{20}{60} \times \frac{1}{12} \times 360\,\text{deg} = 10\,\text{deg}$ ahead of the long hand.
2. The short hand of the clock moves through an angle of 360 degrees in 720 minutes and the long hand moves through an angle of 360 in 60 minutes. We need to find the common time interval T for the short hand to move through ϕ degrees and for the long hand to move through an angle $\phi + 20$ degrees. First we solve for the angle ϕ by equating
$$T = \frac{\phi}{\frac{360}{720}} = \frac{\phi + 20}{\frac{360}{60}}.$$
This yields $\phi = \frac{20}{11}$ degrees and then $T = \frac{40}{11}$ minutes corresponding to the later time four twenty three and thirty-eight seconds.

Problem 2

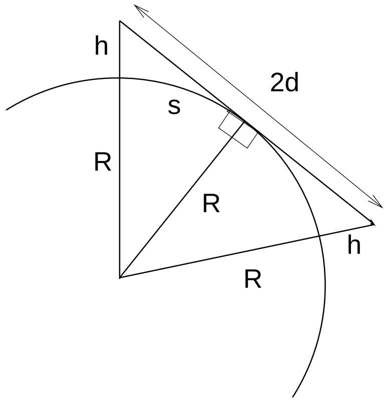
An astronaut plants a flagpole on the surface of the Moon until the top of the flagpole is at eye-level height, at 1.5 metres, and then the astronaut walks off over level ground until the top of the flagpole is just visible on the horizon. The astronaut uses a laser distance meter to measure the straightline distance from his line of sight to the top of the flagpole at 4.6 kilometres. What is the radius of the Moon?

Solution

The geometry (not to scale) is shown in the figure. The flagpole and the astronaut's eye level are both at height h metres, the distance between them is $2d$ in a straightline path, and the radius of the Moon is R.

From Pythagoras's formula we have

$$(h + R)^2 = d^2 + R^2$$



and then rearranging we obtain

$$R = \frac{d^2 - h^2}{2h}.$$

Using the data given we have $h = 1.5$ metres and $2d \approx 4600$ metres so that $R \approx 1.76 \times 10^6$ metres.

Problem 3

At a school dance in the first dancing session all the girls danced but only half the boys danced. Each of the boys that danced partnered exactly two different girls and each of the girls danced with exactly three different boys during the session. In the second dancing session again all girls danced and two-thirds of the boys danced. Each of the boys that danced partnered exactly three different girls.

If each of the girls danced with the same number of different boys how many different boy partners did each girl have in the second session?

Solution

Let b denote the number of boys and g the number of girls. In the first session the total number of dancing partners is

$$n = 2(\frac{b}{2}) = 3g$$

so that the number of boys is $b = 3g$. In the second session the total number of dancing partners is

$$m = 3(\frac{2b}{3}) = xg$$

where x is the number of boys that each girl partners. Thus

$$x = \frac{2b}{g} = \frac{6g}{g} = 6.$$

Problem 4

The zeros of the polynomial

$$P(x) = x^2 + m^2x + n^3$$

are the cubes of the zeros of the polynomial

$$Q(x) = x^2 + mx + n.$$

Find the smallest positive integers m, n for which this can be true.

Solution

Let a, b denote the solutions of

$$x^2 + mx + n = 0.$$

Then a^3, b^3 are the solutions of

$$x^2 + m^2x + n^3 = 0$$

and we also have the relations

$$\begin{aligned} ab &= n, \\ a + b &= -m, \\ a^3b^3 &= n^3, \\ a^3 + b^3 &= -m^2. \end{aligned}$$

But $a^3 + b^3 = (a + b)(a^2 - ab + b^2) = (a + b)((a + b)^2 - 3ab)$ so that $m^2 = m(m^2 - 3n)$. Given that m is a positive integer we now have $m = m^2 - 3n$ and $m(m - 1) = 3n$. The product $m(m - 1)$ must be even so that n must also be even and $n \geq 2$. We also know that 3 must be a divisor of m or $m - 1$ and thus the smallest positive integer m to satisfy this condition is $m = 3$. A simple check now reveals that $n = 2$ and $m = 3$ are the smallest positive integer solutions.

Problem 5

Find all pairs of integers (x, y) such that $6x^2 - 2xy - x + 3y = 22$.

Solution:

First solve for

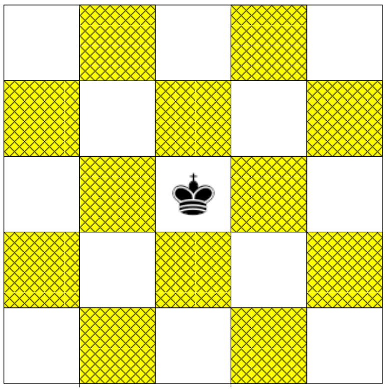
$$\begin{aligned} y &= \frac{6x^2 - x - 22}{2x - 3} \\ &= 3x + 4 - \frac{10}{2x - 3}. \end{aligned}$$

Given that y must be an integer it follows that $2x - 3$ must be a factor of 10. We also know that $2x - 3$ must be an odd integer and so we require $2x - 3 = -1, -5, 1, 5$ resulting in the solutions $(1, 17), (-1, 3), (2, 0), (4, 14)$.

Problem 6

Consider a square array of $(6n - 1) \times (6n - 1)$ squares where n is a positive integer with a king or a knight starting at the central square. The case with $n = 1$ and a king at the central square is shown in the figure.

As in the game of chess suppose that: A king can move one square horizontally, vertically or diagonally in a given move; a knight can move two squares horizontally and one square vertically, or two squares vertically and one horizontally, in a given move.



1. Find the minimum number of moves for the king to get to a corner square.
2. Find the minimum number of moves for the knight to get to a corner square.

Solution:

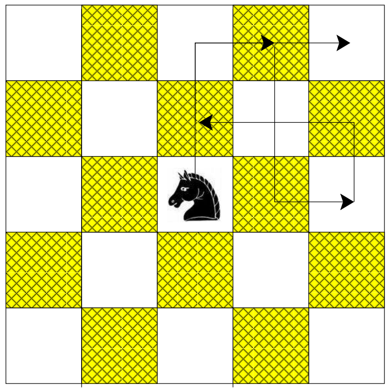
1. Let $(1, 1)$ denote the position of the central square and then $(3n, 3n)$ denotes the position of the right corner square and trivially the required number of moves is $3n - 1$ diagonal moves (through $3n - 1$ diagonal spaces).

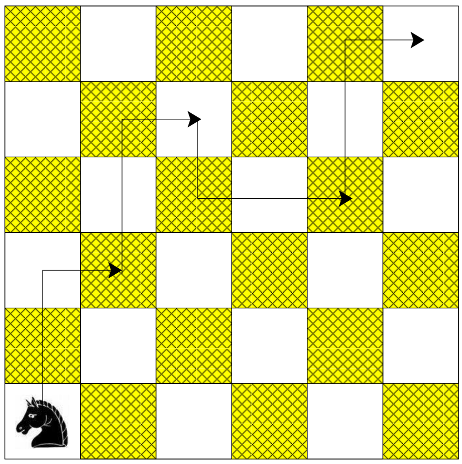
2. The knight can advance at most through $\frac{3}{2}$ diagonal spaces in a single move. Thus to advance through $3n-1$ diagonal spaces requires a minimum of $\frac{2}{3}(3n-1) = 2n-\frac{2}{3}$ moves. But of course a fractional number of moves is not posssible so we have at least $2n$ moves to advance through the $3n - 1$ diagonal spaces. This gives a lower bound on the required number of moves, and the minimum number of moves to move through $3p$ diagonal spaces is $2p$ for any integer p.

   Now consider the case $n = 1$. A valid path for this is shown in the figure below (left). There are other possible four-move paths to get from the centre to the corner square but there are no valid paths with fewer than four moves.

   Now consider the case for $n = 2$. The upper right quadrant with a valid path is shown in the figure below (right). Again there are other possible four-move paths to get from the centre to the corner square and again there are no valid paths with fewer moves.

   To consider the general case, we consider that the knight needs to move through $3n - 1$ diagonal spaces to get from $(1, 1)$ to the corner at $(3n, 3n)$. To keep the knight within the bounds of the array we write $3n - 1 = 3p + r$, where p and r are integers, and the $3p$ diagonal spaces can be advanced through in $2p$ moves. To advance through the remaining r diagonal spaces takes four moves if $r = 2$ (and $p = n - 1$), and four moves if $r = 5$ (and $p = n - 2$). Thus the total number of moves is $2p + 4 = 2n + 2$ if $r = 2$, and $2p + 4 = 2n$ if $r = 5$. Since we have already determined that the number of moves must be at least $2n$ we now have that $2n$ is





the minimum number of moves provided that $n \geq 2$. If $n = 1$ then the minimum number of moves is 4.

Problems and Solutions
Senior Division

Problem 1
See Problem 6 in the Junior Competition.
Solution:
See Problem 6 solution in the Junior Competition.
Problem 2
A box contains three red marbles, two green marbles and one blue marble. In each of three selections one of the marbles in the box is selected at random and replaced with a blue marble. Find the probability that after the three selections there are equal numbers of marbles of each colour.
Solution:
To achieve the desired outcome it is necessary to replace one of the red marbles with a blue marble. There are only three sets of three selections that can achieve this. The sets are:

(i) red, blue, blue

(ii) blue, red, blue

(iii) blue, blue, red

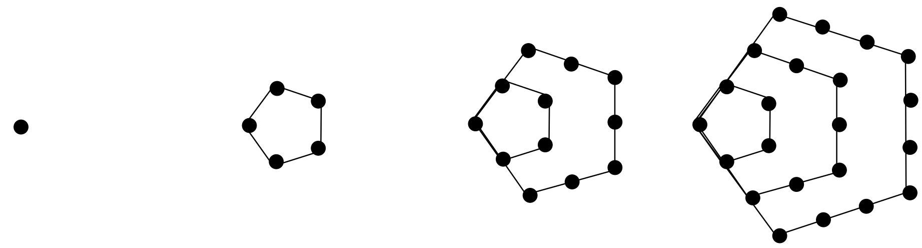
Consider case (i). The probability of selecting the red is $\frac{3}{6} = \frac{1}{2}$. This is now replaced with a blue so that there are now equal numbers of marbles of each colour and the probability of next selecting a blue is $\frac{2}{6} = \frac{1}{3}$. This does not alter the colours of the marbles in the box for the next selection so the probability of again selecting a blue is $\frac{1}{3}$. The combined probability for case (i) is now

$$p_1 = \frac{1}{2}\frac{1}{3}\frac{1}{3} = \frac{1}{18}.$$

Consider case (ii). The probability of selecting the blue is $\frac{1}{6}$. This does not alter the colours of the marbles in the box for the next selection so the probability of selecting a red is $\frac{1}{2}$. There are now equal numbers of marbles of each colour and the probability of next selecting a blue is $\frac{1}{3}$. The combined probability for case (ii) is now

$$p_2 = \frac{1}{6}\frac{1}{2}\frac{1}{3} = \frac{1}{36}.$$

Consider case (iii). The probability of selecting the blue is $\frac{1}{6}$. This does not alter the colours of the marbles in the box for the next selection so the probability of again selecting



a blue is $\frac{1}{6}$ and then the probability of selecting a red is $\frac{1}{2}$. The combined probability for case (ii) is now

$$p_3 = \frac{1}{6}\frac{1}{6}\frac{1}{2} = \frac{1}{72}.$$

Each of the cases will result in equal numbers of each coloured marble giving the total probability

$$p = p_1 + p_2 + p_3 = \frac{4+2+1}{72} = \frac{7}{72}.$$

Problem 3

Pentagonal numbers are polygonal numbers that can be visualized as the numbers of dots in sequences of pentagonal arrays as shown.

The first four pentagonal numbers are

$$P(1) = 1, P(2) = 5, P(3) = 12, P(4) = 22.$$

1. Find $P(2008) - P(2007)$.
2. Triangular numbers are polygonal numbers of the form $T(m) = \frac{1}{2}m(m+1)$ where m is a positive integer. Show that every pentagonal number is one-third of a triangular number.

Solution:

1. It is clear by inspection that consecutive pentagonal numbers are related by

$$P(n+1) = P(n) + 3n + 1$$

from which it follows that

$$P(2008) - P(2007) = 3 \times 2007 + 1 = 6022.$$

2. We can write

$$\begin{aligned} P(n+1) &= 3n + 1 + P(n) \\ &= 3n + 1 + 3(n-1) + 1 + P(n-1) \\ &= 3n + 1 + 3(n-1) + 1 + 3(n-2) + 1 + P(n-2) \\ &= 3n + 1 + 3(n-1) + 1 + 3(n-2) + 1 + P(n-2) + \\ \ldots + 3 + 1 + P(1) \end{aligned}$$

and then using the result that $P(1) = 1$ we deduce that

$$P(n+1) = \sum_{k=0}^{n} 3k + 1.$$

Thus $P(n)$ is the sum of terms in an arithmetic sequence with first term $P(1) = 1$ and common difference $d = 3$ and so using the standard result for arithmetic series we obtain

$$P(n) = nP(1) + \frac{d}{2}(n-1)n$$

which can be expressed as

$$P(n) = n(1 + \frac{3}{2}(n-1)) = \frac{n}{2}(3n-1).$$

Note further that we can write

$$\begin{aligned} P(n) &= \frac{1}{3}\left(\frac{1}{2}3n(3n-1)\right) \\ &= \frac{1}{3}\left(\frac{1}{2}(m+1)m\right) \\ &= \frac{1}{3}T(m) \end{aligned}$$

with $m = 3n - 1$, and so every pentagonal number is one-third of a triangular number.

Problem 4

Let A be the set of all integers of the form $2^r \times 3^s \times 5^t$, where s, r, t are all nonnegative integers.

1. Find the sum of the reciprocals of all integers in the set A.
2. Find the sum of the squares of the reciprocals of all integers in the set A.

Solution:

1. Note that we can write

$$\begin{aligned} \sum_{n \in A} \frac{1}{n} &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{1}{2^r}\frac{1}{3^s}\frac{1}{5^t} \\ &= \sum_{r=0}^{\infty} \frac{1}{2^r} \sum_{s=0}^{\infty} \frac{1}{3^s} \sum_{t=0}^{\infty} \frac{1}{5^t} \end{aligned}$$

Using the standard results for geometric series we have

$$\begin{aligned}\sum_{n\in A}\frac{1}{n} &= (1+\frac{1}{2}+\frac{1}{2^2}+...)(1+\frac{1}{3}+\frac{1}{3^2}+...)(1+\frac{1}{5}+\frac{1}{5^2}+...)\\ &= \frac{1}{1-\frac{1}{2}}\times\frac{1}{1-\frac{1}{3}}\times\frac{1}{1-\frac{1}{5}}\\ &= \frac{15}{4}\end{aligned}$$

2. The second sum can be evaluated in a similar fashion by replacing 2 with 2^2, and 3 with 3^2 and 5 with 5^2. The result is $\frac{25}{16}$.

Problem 5

Let $\mathbb{Q}^+$ denote the set of all positive rational numbers.
Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ with $f(1) = 1$ that satisfy

$$f(x+y)\,(f(x)+f(y)) = f(x)f(y)$$

for all $x, y \in \mathbb{Q}^+$.

Solution:

Let $y = x$, then

$$f(2x)(2f(x)) = f(x)f(x)$$

so that

$$2f(2x) = f(x).$$

Consider $y = 2x$, then

$$f(3x)(f(x)+f(2x)) = f(x)f(2x)$$

and thus $f(3x)(f(x)+\frac{f(x)}{2}) = f(x)\frac{f(x)}{2}$ so that

$$3f(3x) = f(x).$$

It is a simple induction exercise to now show that

$$nf(nx) = f(x)$$

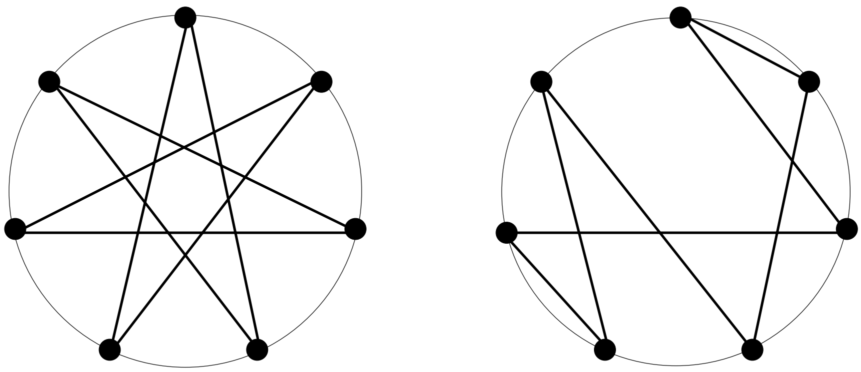
for $n \in \mathbb{N}$.

It follows by writing $x = \frac{m}{n}y$ that

$$nf(my) = f(\frac{m}{n}y)$$

and since $f(m,y) = \frac{1}{m}f(y)$ we also have

$$\frac{n}{m}f(y) = f(\frac{m}{n}y)$$



and hence

$$cf(cy) = f(y)$$

for $c \in \mathbb{Q}$.

Now consider $y = 1$, then

$$cf(c) = f(1) = 1$$

and hence $f(x) = \dfrac{1}{x}$ is the only function that satisfies the given conditions for all $x \in \mathbb{Q}^+$.

Problem 6

Consider $p > 2$, a prime number, and take p distinct points equally spaced on a circle. Join the points in a single line to form a polygon (allowing edges to cross). Each point, except for the starting point, must be visited exactly once. The polygon from this construction is called a *stellated polygon*. Two examples are shown in the figure.

1. Find a formula for the number of different stellated polygons as a function of p.

2. Some of the stellated polygons, such as the one drawn on the left above, are unaltered when rotated through an angle of $\frac{360}{p}^{\circ}$. Such polygons are called regular stellated polygons. The others, such as the one drawn on the right, are non-regular stellated polygons. Find the number of different non-regular stellated polygons as a function of p.

3. Use your answer to (b) to show that

$$\frac{(p-1)! - (p-1)}{2p}$$

   is an integer and hence deduce that p is a factor of $(p-1)! + 1$

Solution:

1. There are p ways to select the starting point, then $p-1$ ways to connect to one of the remaining points, then $p-2$ ways to connect to one of the remaining points and so on. This gives $p!$ paths in total but the number of distinct polygons is less than this. First we divide by p to avoid multiple counting of equivalent polygons produced by paths starting at different points. Of the remaining polygons only half

of them are distinct since a path starting at a given point and its reverse starting at the same point result in the same polygon. Thus we have $\frac{p!}{2p}$ distinct paths.

2. Let the symbol $\triangle(p,k)$ denote the regular stellated polygon with p points and edges, where k indicates that the polygon is formed by connecting each point with the kth point from it in a clockwise direction. With p prime it follows that $\triangle(p,k)$ is a regular stellated polygon for $k = 1, 2, \ldots, p-1$. However, the $\triangle(p,k)$ polygon is equivalent to the $\triangle(p,p-k)$ polygon so the number of distinct regular stellated polygons is $\frac{p-1}{2}$. From this we deduce that the number of non-regular stellated polygons is $\frac{p!}{2p} - \frac{p-1}{2}$.

3. The different non-regular stellated polygons can be grouped into N disjoint classes with p in each class (corresponding to rotations through multiples of $\frac{2\pi}{p}$). Thus we can write
$$\frac{p!}{2p} - \frac{p-1}{2} = Np$$
or equivalently
$$p\left[\frac{(p-1)! - (p-1)}{2p}\right] = Np$$
and thus
$$\left[\frac{(p-1)! - (p-1)}{2p}\right] = N$$
is an integer. A simple rearrangement of the above now yields
$$\left[\frac{(p-1)! + 1}{p}\right] = 2N + 1$$
from which it follows that $(p-1)! + 1$ is divisible by p.