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History of Mathematics: Conventions

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It is strange that it should be so, but it is true all the same, that many of the most debated aspects of Mathematics concern matters that are really completely trivial. They arise from questions of how we make our definitions. Such matters are referred to as *conventions*. Start with a simple example:

Is 1 a prime number?

One very common definition of a prime number is that it is a natural number that has no divisors other than itself and 1. If we use *this* definition then clearly 1 is prime.

Prime numbers also satisfy a congruence relation known as Wilson's Theorem. It goes:

$$(p-1)! = kp - 1, \quad \text{where } k \text{ is a natural number.}$$

Take as an example the number 5. This is a prime and

$$(5-1)! = 24 = 5 \times 5 - 1 \;\; (k = 5)$$

On the other hand, if we had used a composite number (one that *does* have divisors other than itself and 1) in place of the prime p, this would not work. Try it with 6 for example.

$$(6-1)! = 120 = 20 \times 6 \quad \text{(exactly)}.$$

The final -1 is missing!

So what happens if we use 1 in place of p? The value given to $0!$ is 1, and now the Wilson criterion is satisfied, with $k = 2$:

$$(1-1)! = 1 = 2 \times 1 - 1.$$

On the definition just given then, there are two types of natural number:

the primes: $1, 2, 3, 5, 7, \ldots$, and
the composites: $4, 6, 8, 9, 10, \ldots$.

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However, most modern treatments do not take this route. There is a reason for this. A useful theorem (known as the *Fundamental Theorem of Arithmetic*) states that there is exactly *one* set of prime divisors for any natural number. Thus $12 = 2 \times 2 \times 3$, and apart from the trivial matter of listing the prime divisors in a different order, this is the only way to decompose 12 into prime factors. However, if we allow 1 as a prime, then the theorem is false in a trivial but annoying way:

$$12 = 2 \times 2 \times 3 = 1 \times 2 \times 2 \times 3 = 1 \times 1 \times 2 \times 2 \times 3$$

There are *infinitely many* different prime decompositions! To avoid this annoyance, we say that 1 is *not* a prime, and on *this* understanding, all those extra 1s are excluded. The modern convention thus has it that *a* prime number is a natural number *other than* 1 that has no divisors other than itself and 1.

On this definition there are three types of natural numbers:

the primes: $2, 3, 5, 7, \ldots$, the composites: $4, 6, 8, 9, 10, \ldots$, and 1, which sits in a class all of its own.

'But which definition is *right*?', I hear you cry! Well *either*! Just as long as we are clear which definition we are using, then, *as long as we stick to that one*, everything will be all right. The modern definition given just above is now preferred to the alternative because the Fundamental Theorem of Arithmetic is seen as more important than the slight inconvenience of having a special number 1, all in a class of its own.

Another such difference in convention applies to the answer to the question:

Is 0 a natural number?

Most modern accounts say 'yes', but the older convention made the natural numbers synonymous with the counting numbers: $1, 2, 3, \ldots$. There is a lot to be said for this earlier convention. In particular, it makes for direct correspondence with our *psychological* approaches to number. When we first learn to count, we most certainly start with the numeral 1, not 0. Nonetheless, when we come to more sophisticated mathematical notions, it is often simpler to include 0 as a natural number: the *first* natural number. We need then to list 0 among the composite numbers, along with $4,\ 6$ and the rest, because *any* natural number divides with 0. $0 = 0 \times n$, for any natural number n.

A somewhat related case came up when a group of us from Monash University were invited to lead a Mathematics Day at a local high school. One of the activities involved the exploration of the properties of the Fibonacci Sequence. This goes: $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, where each number after the first two is the sum of the two numbers preceding it.

Thus, if F_n is the n^{th} Fibonacci number, then $F_n = F_{n-1} + F_{n-2}$. However, the question arises again: where do we begin the count?

There are several different conventions. The most obvious way is to say: $F_1 = 1,\ F_2 = 1,\ F_3 = 2$, etc. Other accounts start the count at 0, and so have $F_0 = 1,\ F_1 = 1,\ F_2 = 2$, and so on. But several other conventions have also been used and it is one of these that is now most generally adopted. This keeps $F_1 = 1,\ F_2 = 1,\ F_3 = 2$, etc., but also includes the further term $F_0 = 0$. The choice of convention can make a difference (a slight difference) to the formula for F_n . If we follow this last convention, then $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$. This is known as Binet's formula, but it can look slightly different under some of the different conventions. (Sadly, this caused some confusion during the Mathematics Day, with some of us using one convention and others another!) Mathematical Induction (see Parabola, Vol. 44, No. 2) can be used to prove the Binet result; but I leave these details to the reader.

But now let us move on to a more substantial example, one that still generates quite a lot of discussion. What value is to be assigned to the expression 0^0? I wrote on this topic in *Function* back in August 1981, but perhaps I can recapitulate the main points. Here is a summary.

The question of the value of 0^0 is a hardy perennial. There are two conventions, and no confusion need arise if we keep in mind which one is being employed. In one of my own recent papers I used the convention $0^0 = 1$, but in other contexts, I would regard the expression as being undefined in the same sense that $\frac{0}{0}$ is undefined.

The $0^0 = 1$ school has some famous adherents [Leonhard Euler (1707 - 1783), Johann Pfaff (1765 - 1825), August Möbius (1790 - 1868)], while the opposing team [Augustin-Louis Cauchy 1789 - 1857), Guglielmo Libri (1803 - 1869) and two anonymous authors known only as S and 'ein unbekannt' , i.e. anon.] would seem to be rather put in their shade. If we were to 'rank' mathematicians, then the two superstars (Euler and Cauchy) would outshine all the others, but Euler would probably rate above Cauchy. Pfaff and Möbius were both excellent mathematicians (Möbius has a strip named after him!), but Libri was only a relatively minor figure. Nobody seems to know who the others were.

But let us now turn to the actual arguments they used.

Euler noted that $a^0 = 1$ when a is not 0, and said it makes sense to define $a^0 = 1$ also when $a = 0$.

Pfaff and Möbius employed the argument: limit of x^x as x goes to zero is 1.

It is a little difficult to know quite what argument Cauchy used, but we may make an intelligent guess. When $a \neq 0,\ a^0 = a^{b-b} = \frac{a^b}{a^b} = 1$, but when we try to use this same argument in the case of $a = 0$, we end up with $\frac{0}{0}$ which is undefined.

However, matters are not quite so simple. We can easily construct examples in which $\frac{0}{0}$ takes on any value whatsoever, but it is actually quite difficult to construct examples in which 0^0 is not 1 . This is what S and anon. both succeeded in doing; constructing cases in which, as x tends to 0, $f(x)$and $g(x)$ both tend to zero, but $f(x)^{g(x)}$ does not tend to 1. (Cauchy may be read as implying that he had such an example, but, if so, he never divulged it!)

Anon. produced the case $f(x) = x,\ g(x) = \frac{a+x}{\ln x}$. Here it is not difficult to see that $g(x) \to 0$ as $x \to 0$. But $x^{g(x)} = e^{g(x)\ln x} = e^{a+x} \to e^a$ as $x \to 0$. S had a different example: $f(x) = e^{-2/x},\ g(x) = x$. Here it is easy to see that $f(x) \to 0$ as $x \to 0$. But here $f(x)^{g(x)} = e^{-2} \neq 1$. (Actually, S got $e^{-1/2}$. I sincerely hope this was a misprint!)

The difficulty in constructing such examples was first explained by L J Paige in

1954, with slightly fuller accounts by later authors. Here I give the essence of Paige's argument. Consider the function $x^{g(x)}$. This covers anon.'s case exactly, and S's may also be recast in this form, although I omit the details. Now suppose that $g(x)$ is a differentiable function satisfying $g(0) = 0$. Then let $g'(0) = k$, say. It follows that for small values of $x, g(x) \approx kx$, and so $x^{g(x)} \approx x^{kx} = (x^x)^k \to 1^k = 1$ as a consequence of the Pfaff-Möbius result.

So, it would seem that the less-favoured Cauchy team won the day. Certainly they had all the running for many years. When the project was commenced of collecting all Euler's works together, his discussion of 0^0 was placed in the very first volume, but the twentieth-century editor felt compelled to insert a footnoted disclaimer.

However, this is not the end of the story. In 1970, a brief note appeared in the journal *The Mathematics Teacher* arguing for the convention $0^0 = 1$. The author was the mathematics educator Herbert E Vaughan. Vaughan gave a number of examples in which this convention proves useful. Here is one (his first, it is actually a better-expressed rerun of one by Libri, whose own contributions to the debate are rather tedious and muddled). (It uses the formula for the sum of a geometric progression.)

$$1 = \frac{1}{1-0} = 0^0 + 0^1 + 0^2 + ... = 0^0 + 0 + 0 + ... = 0^0.$$

The case was taken up by the influential contemporary mathematician Donald Knuth, who holds that the binomial theorem (just used above) is too important to let go. If we consider the competing claims of $\lim_{x\to 0} x^0 = 1$ and $\lim_{x\to 0} 0^x = 0$ or ∞, the first is more important. He and now others also have urged that computer defaults be set to reflect the Euler convention, rather than the Cauchy. Some years ago, I tested how far their pleas had got. My three hand-held calculators, Maple V (Version 5.1) and Excel all gave error messages when I asked them to evaluate 0^0. More modern calculators may be different; Excel (2003) still wouldn't calculate a value, but both Matlab (R2006b) and Scientific Workplace (Version 5) return the value 1.

So the earlier convention is beginning to prevail.