Problems 1301–1310

Q1301 (Suggested by J. Guest, Victoria)

Solve the quartic
$$(x+1)(x+5)(x-3)(x-7) = -135.$$

Q1302 Let α, β and γ be the angles of one triangle, and α', β' and γ' be the angles of another triangle. Assume that $\alpha = \alpha'$, $\beta \geq \gamma$ and $\beta' \geq \gamma'$. Prove that

$$\sin\alpha + \sin\beta + \sin\gamma \geq \sin\alpha' + \sin\beta' + \sin\gamma'$$

if and only if
$$\beta - \gamma \leq \beta' - \gamma'.$$

Q1303 (Suggested by Dr. Panagiote Ligouras, Leonardo da Vinci High School, Noci, Bari, Italy. Edited.)

Let m_a, m_b, m_c be the medians, h_a, h_b, h_c the heights, l_a, l_b, l_c the bisectors and R the circumradius of a scalene triangle ABC. Prove that

$$\frac{l_a^3(m_a-h_a)\sqrt{m_a h_a}}{h_a(l_a^2-h_a^2)} + \frac{l_b^3(m_b-h_b)\sqrt{m_b h_b}}{h_b(l_b^2-h_b^2)} + \frac{l_c^3(m_c-h_c)\sqrt{m_c h_c}}{h_c(l_c^2-h_c^2)} < 6R^2.$$

Q1304 Prove that the equation $x^2 - 2y^2 = 5$ has no integral roots.

Q1305 The result in **Q1304** is also true in a more general case with the right-hand side being $m = 8k+3$ or $m = 8k-3$, $k = 1, 2, \ldots$. Prove this!

Q1306 Find all positive integers n such that $2^n + 1$ is a multiple of 3.

Q1307 Let a, b, c and d be, respectively, the lengths of the sides AB, BC, CD, and DA of a quadrilateral $ABCD$. Prove that if S is the area of $ABCD$ then

$$S \leq \frac{a+c}{2} \times \frac{b+d}{2}.$$

When does equality occur?

Q1308 In a triangle ABC let H be the foot of the altitude from A, and M be the midpoint of BC. On the circumcircle, let D be the midpoint of the arc BC which does not contain A. Assume that there exists a point I on the edge BC satisfying $IB \times IC = IA^2$. Prove that $AH \leq MD$. Is the converse true?

Q1309 Assume that there exists a point I on the side BC of a triangle ABC which satisfies $IA^2 = IB \times IC$. Prove that

$$\sin B \times \sin C \leq \sin^2 \frac{A}{2}.$$

Is the converse true?

Q1310 Let a, b, c, and d be 4 positive real numbers satisfying

$$\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+d^4} = 1.$$

Prove that $abcd \geq 3$.