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Solutions to Problems 1301–1310

Q1301 (Suggested by J. Guest, Victoria) Solve the quartic

$$(x+1)(x+5)(x-3)(x-7) = -135.$$

ANS: (suggested by David Shaw, Geelong, Victoria)

Rearrange the equation as

$$(x^2-2x-3)(x^2-2x-35) = -135.$$

By setting $z = x^2 - 2x$, the above equation becomes

$$z^2 - 38z + 240 = 0,$$

which has solutions $z_1 = 30$ and $z_2 = 8$. Solving the two equations $x^2 - 2x = 30$ and $x^2 - 2x = 8$ results in 4 solutions to the quartic equation

$$x_1 = 1+\sqrt{31}, \quad x_2 = 1-\sqrt{31}, \quad x_3 = 4, \quad \text{and} \quad x_4 = -2.$$

Q1302 Let α, β and γ be the angles of one triangle, and α', β' and γ' be the angles of another triangle. Assume that $\alpha = \alpha'$, $\beta \geq \gamma$ and $\beta' \geq \gamma'$. Prove that

$$\sin\alpha + \sin\beta + \sin\gamma \geq \sin\alpha' + \sin\beta' + \sin\gamma'$$

if and only if

$$\beta - \gamma \leq \beta' - \gamma'.$$

ANS: Assume that

$$\sin\alpha + \sin\beta + \sin\gamma \geq \sin\alpha' + \sin\beta' + \sin\gamma'.$$

Then since $\alpha = \alpha'$ we have

$$\sin\beta + \sin\gamma \geq \sin\beta' + \sin\gamma'.$$

By using the addition formula for sines and cosines we can prove that

$$\sin\beta + \sin\gamma = 2\sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}$$

and

$$\sin\beta' + \sin\gamma' = 2\sin\frac{\beta'+\gamma'}{2}\cos\frac{\beta'-\gamma'}{2}.$$

Noting that the sum of the angles in a triangle is $180°$ we deduce

$$\cos\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2}\geq\cos\frac{\alpha'}{2}\cos\frac{\beta'-\gamma'}{2}.$$

or

$$\cos\frac{\beta-\gamma}{2}\geq\cos\frac{\beta'-\gamma'}{2}.$$

The fact that $0\leq\frac{\beta-\gamma}{2},\frac{\beta'-\gamma'}{2}\leq 90°$ gives $\frac{\beta-\gamma}{2}\leq\frac{\beta'-\gamma'}{2}$, therefore $\beta-\gamma\leq\beta'-\gamma'$. By reversing the argument, we can prove that if $\beta-\gamma\leq\beta'-\gamma'$ then $\sin\alpha+\sin\beta+\sin\gamma\geq\sin\alpha'+\sin\beta'+\sin\gamma'$.

Q1303 (Suggested by Dr. Panagiote Ligouras, Leonardo da Vinci High School, Noci, Bari, Italy. Edited.)

Let m_a, m_b, m_c be the medians, h_a, h_b, h_c the heights, l_a, l_b, l_c the bisectors and R the circumradius of a scalene triangle ABC. Prove that

$$\frac{l_a^3(m_a-h_a)\sqrt{m_ah_a}}{h_a(l_a^2-h_a^2)}+\frac{l_b^3(m_b-h_b)\sqrt{m_bh_b}}{h_b(l_b^2-h_b^2)}+\frac{l_c^3(m_c-h_c)\sqrt{m_ch_c}}{h_c(l_c^2-h_c^2)}<6R^2.$$

ANS: Since ΔABC is scalene, $h_a<l_a$, $h_b<l_b$ and $h_c<l_c$. Hence

$$\begin{aligned}&\frac{l_a^3(m_a-h_a)\sqrt{m_ah_a}}{h_a(l_a^2-h_a^2)}+\frac{l_b^3(m_b-h_b)\sqrt{m_bh_b}}{h_b(l_b^2-h_b^2)}+\frac{l_c^3(m_c-h_c)\sqrt{m_ch_c}}{h_c(l_c^2-h_c^2)}\\&<\frac{l_a^4(m_a-h_a)\sqrt{m_ah_a}}{h_a^2(l_a^2-h_a^2)}+\frac{l_b^4(m_b-h_b)\sqrt{m_bh_b}}{h_b^2(l_b^2-h_b^2)}+\frac{l_c^4(m_c-h_c)\sqrt{m_ch_c}}{h_c^2(l_c^2-h_c^2)}.\end{aligned}$$

By using $\sqrt{\alpha\beta}\leq(\alpha+\beta)/2$ we deduce

$$\begin{aligned}&\frac{l_a^3(m_a-h_a)\sqrt{m_ah_a}}{h_a(l_a^2-h_a^2)}+\frac{l_b^3(m_b-h_b)\sqrt{m_bh_b}}{h_b(l_b^2-h_b^2)}+\frac{l_c^3(m_c-h_c)\sqrt{m_ch_c}}{h_c(l_c^2-h_c^2)}\\&<\frac{1}{2}\frac{l_a^4(m_a-h_a)(m_a+h_a)}{h_a^2(l_a^2-h_a^2)}+\frac{1}{2}\frac{l_b^4(m_b-h_b)(m_b+h_b)}{h_b^2(l_b^2-h_b^2)}+\frac{1}{2}\frac{l_c^4(m_c-h_c)(m_c+h_c)}{h_c^2(l_c^2-h_c^2)}\\&=\frac{1}{2}\frac{l_a^4(m_a^2-h_a^2)}{h_a^2(l_a^2-h_a^2)}+\frac{1}{2}\frac{l_b^4(m_b^2-h_b^2)}{h_b^2(l_b^2-h_b^2)}+\frac{1}{2}\frac{l_c^4(m_c^2-h_c^2)}{h_c^2(l_c^2-h_c^2)}.\end{aligned}$$

If we can prove that

$$\frac{l_a^4(m_a^2-h_a^2)}{h_a^2(l_a^2-h_a^2)}=\frac{l_b^4(m_b^2-h_b^2)}{h_b^2(l_b^2-h_b^2)}=\frac{l_c^4(m_c^2-h_c^2)}{h_c^2(l_c^2-h_c^2)}=4R^2,$$

then the required inequality is proved.

It suffices to prove

$$\frac{l_a^4(m_a^2-h_a^2)}{h_a^2(l_a^2-h_a^2)}=4R^2.$$

Let H, L and M be the feet of h_a, l_a, and m_a on BC. The line AL cuts the circumcircle again at N. Since ΔABN is similar to ΔALC (having $\angle BAN = \angle LAC$ and $\angle ANB = \angle ACL$) we have

$$\frac{AC}{AL} = \frac{AN}{AB} \quad \text{or} \quad AN^2 = \frac{b^2c^2}{l_a^2}. \tag{0.1}$$

On the other hand, since ΔALH is similar to ΔNLM (check this!) we have

$$\frac{NL}{AL} = \frac{ML}{LH},$$

implying

$$\frac{AN}{AL} = \frac{MH}{LH} \quad \text{or} \quad AN^2 = \frac{l_a^2(m_a^2 - h_a^2)}{l_a^2 - h_a^2}. \tag{0.2}$$

(0.1) and (0.2) give

$$b^2c^2 = \frac{l_a^4(m_a^2 - h_a^2)}{l_a^2 - h_a^2}.$$

It is well known that $R = \frac{bc}{2h_a}$. Therefore,

$$\frac{l_a^4(m_a^2 - h_a^2)}{h_a^2(l_a^2 - h_a^2)} = 4R^2.$$

Q1304 Prove that the equation $x^2 - 2y^2 = 5$ has no integral roots.

ANS: Assume that there exist integers x and y satisfying the given equation. It follows that

$$(x-1)(x+1) = 2y^2 + 4.$$

Thus $x - 1$ and $x + 1$ are two consecutive even integers. By writing $x - 1 = 2n$ and $x + 1 = 2n + 2$ for some integer n we deduce

$$y^2 + 2 = 2n(n+1),$$

implying that y is an even integer. Putting $y = 2m$ and substituting back into the above equation we obtain

$$2m^2 + 1 = n(n+1),$$

which is a contradiction, because the left-hand side is odd whereas the right-hand side is even.

Q1305 The result in **Q1304** is also true in a more general case with the right-hand side being $m = 8k + 3$ or $m = 8k - 3$, $k = 1, 2, \ldots$. Prove this!

ANS: We prove only the case when $m = 8k + 3$. Similarly to **Q1304** we now have

$$y^2 + 4k + 1 = 2n(n+1).$$

Since $n(n+1)$ is even, we have

$$y^2 + 4k + 1 = 4l$$

for some positive integer l.

Q1306 Find all positive integers n such that $2^n + 1$ is a multiple of 3.

ANS:

Solution 1: (suggested by David Shaw, Geelong, Victoria)

Since $2 \equiv -1 \pmod 3$ we have

$$2^n + 1 \equiv (-1)^n + 1 \pmod 3 \equiv \begin{cases} 0 \pmod 3 & \text{if } n \text{ is odd} \\ 2 \pmod 3 & \text{if } n \text{ is even.} \end{cases}$$

Hence $2^n + 1$ is a multiple of 3 if and only if n is an odd integer.

Solution 2: By using

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

with $a = 2$ and $b = -1$ we obtain

$$2^n - (-1)^n = 3A, \tag{0.3}$$

where A is an integer. If n is odd we deduce from (0.3)

$$2^n + 1 = 3A,$$

that is $2^n + 1$ is a multiple of 3. If n is even we deduce from (0.3)

$$2^n - 1 = 3A,$$

and thus $2^n + 1 = 3A + 2$. Therefore, $2^n + 1$ is a multiple of 3 if and only if n is odd.

Q1307 Let a, b, c and d be, respectively, the lengths of the sides AB, BC, CD, and DA of a quadrilateral $ABCD$. Prove that if S is the area of $ABCD$ then

$$S \le \frac{a+c}{2} \times \frac{b+d}{2}.$$

When does the equality occur?

ANS: We consider two cases:

Case 1: $ABCD$ is convex. Then

$$S = S_{\Delta ABD} + S_{\Delta BCD} = \frac{1}{2}(ad\ \sin A + bc\ \sin C) \le \frac{1}{2}(ad + bc).$$

Similarly, $S \le \frac{1}{2}(cd + ab)$. Hence

$$2S \le \frac{1}{2}(ad + bc + cd + ab) = \frac{1}{2}(a+c)(b+d),$$

which then implies the required inequality.

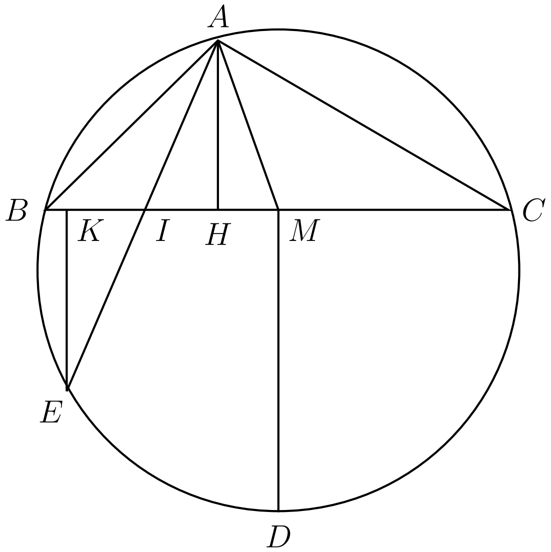
Case 2: $ABCD$ is not convex. Then one diagonal is outside the quadrilateral. Assume that this diagonal is BD. Let C' be the reflection of C about BD. Then $ABC'D$ is convex and has side lengths a, b, c and d. Therefore, it follows from Case 1 that

$$S_{ABCD} < S_{ABC'D} \leq \frac{a+c}{2} \times \frac{b+d}{2}.$$

Equality occurs when $\sin A = \sin B = \sin C = \sin D = 90^\circ$. In that case $ABCD$ is a rectangle ($a = c$ and $b = d$) and $S = ab$.

Q1308 In a triangle ABC let H be the foot of the altitude from A, and M be the midpoint of BC. On the circumcircle, let D be the midpoint of the arc BC which does not contain A. Assume that there exists a point I on the edge BC satisfying $IB \times IC = IA^2$. Prove that $AH \leq MD$. Is the converse true?

ANS:



Prolong AI to cut the circle at E and draw $EK \perp BC$ as in the picture. Then $IA \times IE = IB \times IC$. Hence $IA = IE$, and therefore $\Delta AHI = \Delta EKI$. It follows that $AH = EK \leq MD$.

Now assume that $AH \leq MD$. We show that there exists a point I on BC satisfying $IB \times IC = IA^2$. Let F be the point on MD such that $MF = AH$. Draw a line passing through F and parallel with BC. This line cuts the circle at two points (E is one of these two points). Connecting A with any one of these two points, the intersection with BC will be I satisfying $IB \times IC = IA^2$. Check this!

Q1309 Assume that there exists a point I on the side BC of a triangle ABC which satisfies $IA^2 = IB \times IC$. Prove that

$$\sin B \times \sin C \leq \sin^2 \frac{A}{2}.$$

Is the converse true?

ANS: First we note that

$$AB = 2R \sin C, \quad BC = 2R \sin A, \quad \text{and} \quad CA = 2R \sin B, \tag{0.4}$$

where R is the radius of the circle in **Q1308**. For example, to prove $BC = 2R \sin A$ we note that if O is the centre of the circle then

$$BC = 2BM = 2OB \sin(\angle BOM) = 2R \sin A.$$

Similarly, we can prove the other identities. Since $AH = AB \sin B$ we have

$$AH = 2R \sin B \times \sin C. \tag{0.5}$$

On the other hand, we have

$$MD = MB \cot(\angle BDM) = \frac{1}{2} BC \cot(\angle BDC/2) = \frac{1}{2} BC \tan(A/2).$$

By using (0.4) we deduce

$$MD = R \sin A \times \tan(A/2) = 2R \sin^2(A/2).$$

The required result now follows from (0.5) and the result in **Q1308**. Check that the converse is also true, that is if $\sin B \times \sin C \leq \sin^2(A/2)$ then there exists I on BC satisfying $IA^2 = IB \times IC$.

Q1310 Let a, b, c, and d be 4 positive real numbers satisfying

$$\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+d^4} = 1.$$

Prove that $abcd \geq 3$.

ANS: By writing

$$a^2 = \tan A, \quad b^2 = \tan B, \quad c^2 = \tan C, \quad d^2 = \tan D,$$

where $A, B, C, D \in (0, \pi/2)$, we have from the given identity

$$\sin^2 A = 1 - \cos^2 A = \cos^2 B + \cos^2 C + \cos^2 D.$$

The Cauchy inequality

$$\frac{\alpha + \beta + \gamma}{3} \geq \sqrt[3]{\alpha\beta\gamma}$$

yields

$$\sin^2 A \geq 3(\cos^2 B \ \cos^2 C \ \cos^2 D)^{2/3}.$$

Similarly, we have

$$\sin^2 B \geq 3(\cos^2 C \ \cos^2 D \ \cos^2 A)^{2/3}$$
$$\sin^2 C \geq 3(\cos^2 D \ \cos^2 A \ \cos^2 B)^{2/3}$$
$$\sin^2 D \geq 3(\cos^2 A \ \cos^2 B \ \cos^2 C)^{2/3}.$$

Multiplying all four inequalities gives

$$\sin^2 A \ \sin^2 B \ \sin^2 C \ \sin^2 D \geq 3^4 \cos^2 A \ \cos^2 B \ \cos^2 C \ \cos^2 D,$$

implying

$$\tan^2 A \ \tan^2 B \ \tan^2 C \ \tan^2 D \geq 3^4,$$

or $abcd \geq 3$.