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Mathematical Curiosities: Pentagonal Identities

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Polygonal numbers enumerate the number of points in a regular geometrical arrangement of the points in the shape of a regular polygon. An example is the *triangular number* T_n which enumerates the number of points in a regular triangular lattice of points whose overall shape is a triangle. The geometrical arrangement leading to the first five triangular numbers is shown below.

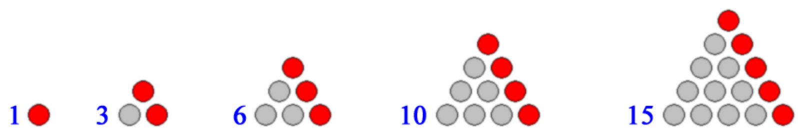

Figure 1: Geometrical arrangement for triangular numbers. (Image from Wikimedia Commons.)

From the geometrical arrangement it can be seen that in general

$$T_n = \sum_{k=1}^{n} k = \frac{1}{2}n(n+1).$$

Square numbers S_n can be defined in a similar fashion enumerating the number of points in a regular square lattice of points whose overall shape is a square. By construction $S_n = n^2$. The geometrical arrangement is shown below and it can be seen that $S_n = \sum_{k=1}^{n}(2k-1)$.

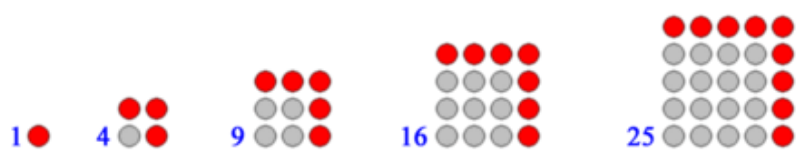

Figure 2: Geometrical arrangement for square numbers. (Image from Wikimedia Commons.)

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The geometrical construction for the first four *pentagonal numbers* is shown in the figure below.

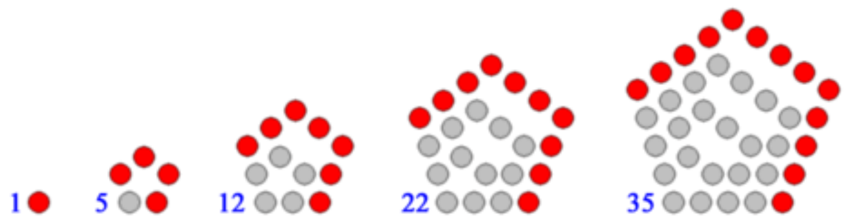

Figure 3: Geometrical arrangement for pentagonal numbers. (Image from Wikimedia Commons.)

It is easy to identify the geometrical pattern leading to the equivalence

$$P_n = \sum_{k=1}^{n} 3k - 2 = \frac{1}{2}(3n^2 - n).$$

Hexagonal numbers, heptagonal numbers etc. can also be defined. In general if s denotes the number of sides of a regular polygon then the nth s-gonal number is given by

$$\theta(s,n) = \frac{1}{2}(n^2(s-2) - n(s-4)).$$

For example, if $s = 5$ then $\theta(5,n) = \frac{1}{2}(3n^2 - n) = P_n$. There have been a large number of identities and theorems proved for polygonal numbers. Perhaps the most famous is Fermat's polygonal number theorem which states that every positive integer can be written as the sum of s or fewer s-gonal numbers. For example, 17 can be written as the sum of three triangular numbers, $T_1 + T_3 + T_4$, two square numbers, $S_1 + S_4$, and two pentagonal numbers, $P_2 + P_3$.

A curiosity involving pentagonal numbers and concatenation

A curiosity considered here is that the sum of two polygonal numbers $\theta(s,n)$ and $\theta(s,m)$ is sometimes equal to the number composed of the concatenation of n and m. For example, among the pentagonal numbers, $P_3 = 12$ and $P_4 = 22$, so that $P_3 + P_4 = 34$, which is the concatenation of 3 and 4. We will refer to such pairs as *polycat pairs*. Similar patterns can be found among other polygonal numbers, but they are particularly plentiful among the pentagonal numbers. The table on the next page shows many more examples.

	(n,m)	$\theta(s,n)$	$\theta(s,m)$	$\theta(s,n)+\theta(s,m)$
$s=3$	(19,1)	190	1	191
	(90,415)	4095	86320	90415
	(585,910)	171405	414505	585910
$s=4$	(10,1)	100	1	101
	(10,100)	100	10000	10100
	(12,33)	144	1089	1233
$s=5$	(1,3)	1	12	13
	(3,4)	12	22	34
	(4,4)	22	22	44
$s=6$	(4,3)	28	15	43
	(12,22)	276	946	1222
	(16,24)	496	1128	1624

It is easy to write computer programs to construct examples of polycat pairs. With a little thought it is also possible to construct examples without an exhaustive search. It is left as an exercise for the reader to show that

$$\left(\frac{1}{3}(10^k-1),\ \frac{1}{3}(10^k+2)\right)$$

is a pentagonal polycat pair for all positive integers k.