**Parabola Volume 52, Issue 1 (2016)**

# 1 Solutions 1481–1490

**Q1481** Prove that if the denominator $q$ of a fraction $p/q$ is the number consisting of $n$ digits, all equal to 9, and if $p$ is less than $q$, then $p/q$ can be written as a repeating decimal in which the repeating part has length $n$ and contains the digits of $p$, preceded by a sufficient number of 0s to give that length.

**SOLUTION** We can write down the repeating decimal described and sum a geometric series to get

$$\frac{p}{10^n} + \frac{p}{10^{2n}} + \frac{p}{10^{3n}} + \cdots = \frac{p}{10^n}\frac{1}{1-\frac{1}{10^n}} = \frac{p}{10^n - 1} = \frac{p}{q} .$$

**Q1482** If the acute angles $\alpha$ and $\beta$ satisfy the equation

$$(1 - \cot\alpha)(1 - \cot\beta) = 2 ,$$

find the value of $\alpha + \beta$.

**SOLUTION** Multiplying by $\sin\alpha\sin\beta$, we have

$$(\sin\alpha - \cos\alpha)(\sin\beta - \cos\beta) = 2\sin\alpha\sin\beta ,$$

which can be arranged to give

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta = \sin\alpha\cos\beta + \cos\alpha\sin\beta .$$

Using the formulae for sine and cosine of a sum, we have

$$\cos(\alpha + \beta) = \sin(\alpha + \beta)$$

and hence

$$\tan(\alpha + \beta) = 1 .$$

Thus $\alpha + \beta = \frac{\pi}{4} + n\pi$, and since $\alpha$ and $\beta$ are acute, we have $0 < \alpha + \beta < \pi$ and hence

$$\alpha + \beta = \frac{\pi}{4} .$$

**Q1483** We have a geometric sequence $a_1, a_2, a_3, \ldots$ for which it is known that

$$a_1 + a_3 + a_5 + a_7 = 5102 \quad \text{and} \quad a_2 + a_4 + a_6 = 2015 .$$

Evaluate

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 .$$

**SOLUTION** Since we have a geometric sequence, each term is the geometric mean of the adjacent terms:

$$a_2^2 = a_1 a_3 \ , \quad a_3^2 = a_2 a_4$$

and so on. The same sort of thing holds if we move not one step but, say, $k$ steps away from a particular term,

$$a_n^2 = a_{n-k} a_{n+k} \ .$$

Therefore

$$\begin{aligned}
(a_1 + a_3 + a_5 + a_7)^2 &= a_1^2 + a_3^2 + a_5^2 + a_7^2 + 2a_1a_3 + 2a_1a_5 \\
&\qquad + 2a_1a_7 + 2a_3a_5 + 2a_3a_7 + 2a_5a_7 \\
&= a_1^2 + a_3^2 + a_5^2 + a_7^2 + 2a_2^2 + 2a_3^2 \\
&\qquad + 2a_4^2 + 2a_4^2 + 2a_5^2 + 2a_6^2 \\
&= a_1^2 + 2a_2^2 + 3a_3^2 + 4a_4^2 + 3a_5^2 + 2a_6^2 + a_7^2 \\
(a_2 + a_4 + a_6)^2 &= a_2^2 + a_4^2 + a_6^2 + 2a_2a_4 + 2a_2a_6 + 2a_4a_6 \\
&= a_2^2 + a_4^2 + a_6^2 + 2a_3^2 + 2a_4^2 + 2a_5^2 \\
&= a_2^2 + 2a_3^2 + 3a_4^2 + 2a_5^2 + a_6^2 \ .
\end{aligned}$$

Subtracting these two expressions gives exactly what we want!

$$\begin{aligned}
a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 &= (a_1 + a_3 + a_5 + a_7)^2 - (a_2 + a_4 + a_6)^2 \\
&= 5102^2 - 2015^2 \\
&= 21970179 \ .
\end{aligned}$$

**Q1484** Let $n = 6$ and take the numbers in the $n$th row of Pascal's triangle, leaving out the last of them:

$$1,\ 6,\ 15,\ 20,\ 15,\ 6 \, .$$

Notice that in this list the first number is a multiple of $1$, the second is a multiple of $2$, the third is a multiple of $3$ and so on, all the way through to the $n$th number, which is a multiple of $n$. Prove that this works whenever $n + 1$ is prime.

**SOLUTION** Writing the claim in algebraic terms, we are asked to prove that if $n + 1$ is prime and $k = 1, 2, \ldots, n$, then

$$\binom{n}{k-1} \quad \text{is a multiple of} \quad k \ .$$

Now according to a reasonably well known formula (proof below),

$$k\binom{n+1}{k} = (n+1)\binom{n}{k-1} \ .$$

Since binomial coefficients are integers, this shows that the right-hand side is a multiple of $k$. But if $n + 1$ is prime and $k = 1, 2, \ldots, n$, then $k$ and $n + 1$ have no common factor, so $\binom{n}{k-1}$ is a multiple of $k$, as claimed.

**Addendum**: proof of the above formula.

*Method 1*, a combinatorial proof. Consider a club with $n+1$ members. We wish to choose $k$ members to form a committee, and one of those $k$ to be the president of the committee. To do this we could choose the whole committee first, so $\binom{n+1}{k}$ possibilities; then there are $k$ ways to choose the president, so there are $k\binom{n+1}{k}$ possible outcomes. On the other hand, we could choose the president first, so $n+1$ possibilities; then choose $k-1$ further committee members from the remaining $n$ club members; so $(n+1)\binom{n}{k-1}$ possible outcomes. Since the two expressions we have just found are answers to the same problem, they must be equal.

*Method 2*, an algebraic proof: write both sides in terms of factorials and simplify. We leave this to the reader.

**Q1485** How many 10-digit numbers $x$ are there such that $x$ ends with the digits 2015 and $x^2$ begins with the digits 2015?

**SOLUTION** If $x$ is a 10-digit number, then $10^9 \le x < 10^{10}$ and so

$$10^{18} \le x^2 < 10^{20} .$$

If $x^2$ begins with the digits 2015, there are two options:

$$\begin{aligned} & 2015 \times 10^{15} \le x^2 < 2016 \times 10^{15} \\ \text{or} \quad & 2015 \times 10^{16} \le x^2 < 2016 \times 10^{16} . \end{aligned}$$

Using a calculator (and remembering that $x$ is an integer),

$$\begin{aligned} & 1419506957 \le x \le 1419859147 \\ \text{or} \quad & 4488875138 \le x \le 4489988864 . \end{aligned}$$

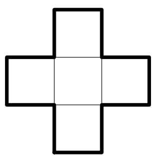
But since $x$ ends in the digits 2015, the options are

$$x = 1419512015 \text{ to } 1419852015 \quad \text{or} \quad 4488882015 \text{ to } 4489982015 ,$$

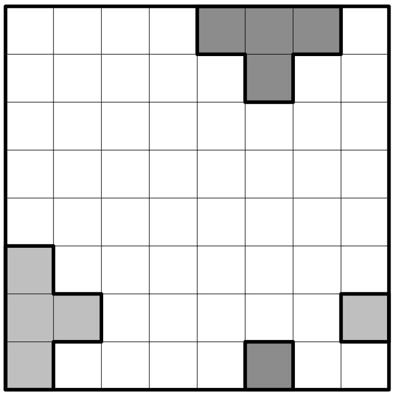
and the number of possibilities is

$$(141985 - 141950) + (448998 - 448887) = 146 .$$

**Q1486** In Question 1479 we showed that a maximum of eight crosses like this one

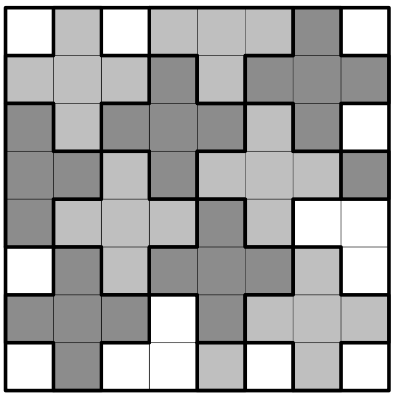


can be placed without overlapping on an $8 \times 8$ chessboard. Next, imagine that the top of the board is joined to the bottom and the left side is joined to the right, so that crosses may be placed (for example) as shown, in addition to the "ordinary" placements.



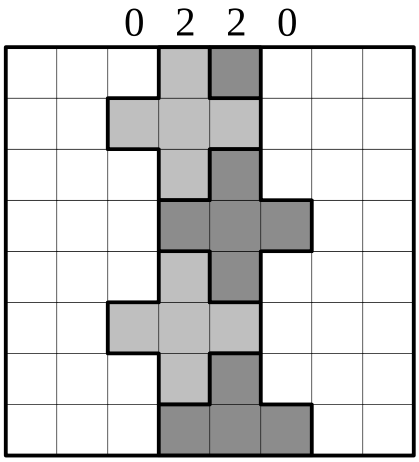
What is now the maximum number of crosses that can be placed on the board?

**SOLUTION** The maximum is $10$. This can quite easily be done by trial and error,



but once again the difficult part of the question is to prove that more than this number is not possible.

Each cross has a centre square. We shall consider the number of "centres" in each column: for example, the above diagram gives the pattern $12112120$. Note that since the left and right sides of the board are regarded as being adjacent, we could just as well describe the pattern as $12012112$ (starting at the third column from the right). We begin with a simple observation: each column of the board can contain at most two "centres" (for if there were three, those centres plus the squares above and below them would require $9$ squares in the column). Next: can we have two adjacent columns containing two centres each? Yes we can, but then each of these columns contains two centres and four squares above and below them from the same crosses, plus two squares from the crosses with centres in the next column. This fills up all $8$ squares in each of these columns,




and it follows that the two columns to the left and right cannot contain any centres.

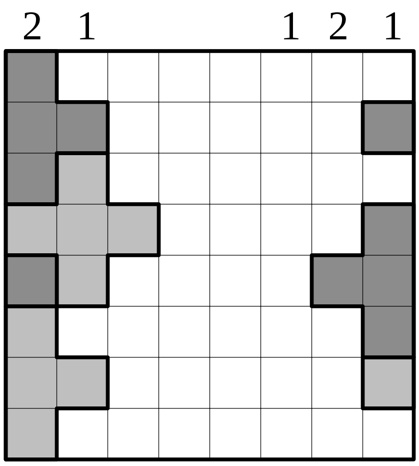
Now suppose that it is possible to place 11 crosses on the "wraparound" chessboard. Then there must be four adjacent columns which contain, collectively, 6 or more centres. The possible patterns with 8 or 7 centres are

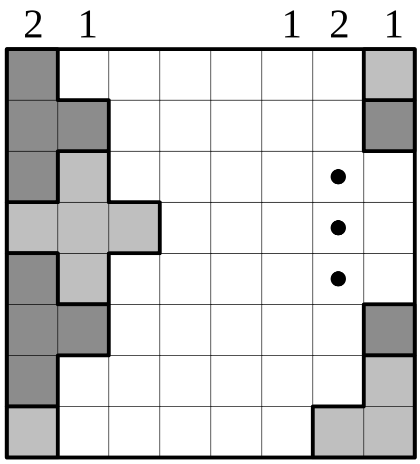
$$2222\,,\quad 2221\,,\quad 2212\,,$$

and since we have already proved that two adjacent 2s must be surrounded by a 0 on each side, these are all impossible. Thus there must be 6 centres in four adjacent columns, and 5 centres in the other four. In fact, we can never have a pair of adjacent 2s at all: for then four consecutive columns would be 0220, and the other four columns would contain seven centres, which is impossible. Because of this, the only possible pattern of six centres in four adjacent columns is 2121. This must extend to a pattern

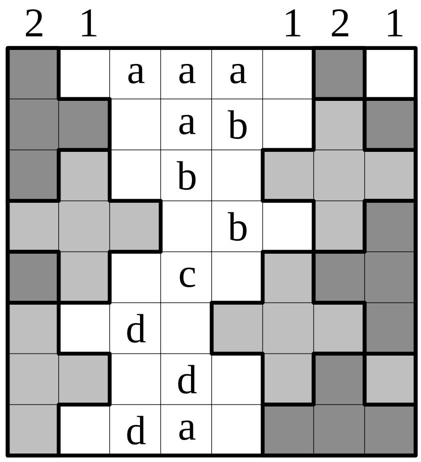
$$21212120 \quad \text{or} \quad 21212111 \quad \text{or} \quad 21211211$$

for the whole board. Each of these contains a sub-pattern 12121, and this is what we investigate next. For a column with two centres, there are two options: the crosses with the centres in this column are adjacent to each other, or they are not. These are illustrated in the following diagrams,

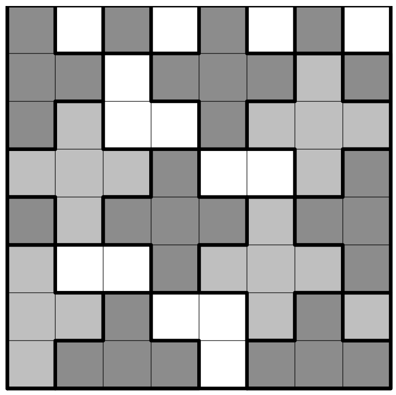


Now in the second diagram there must be two centres in the squares marked with dots: it is easy to see that this is impossible without the crosses overlapping. So this case is ruled out. The first diagram can be continued to the following:




To avoid overlapping we can have only one centre marked $a$, only one marked $b$ and only one marked $d$: so if we are to place four further crosses, the single centre marked $c$ is obligatory. This eliminates two of the possible $b$s and $d$s, forcing us to place ten crosses as follows,



and there is no room for an eleventh.

**Q1487** Determine how many values of $x$ satisfy the conditions

$$x^2 - x\lfloor x \rfloor = 20{\cdot}15 \,, \qquad x \le 2015 \,.$$

Here the notation $\lfloor x \rfloor$ denotes $x$ rounded to the nearest integer downwards, for example, $\lfloor \pi \rfloor = 3$.

**SOLUTION** First we note that $x = 2015$ is not a solution, so we may assume $x < 2015$. Now let $\lfloor x \rfloor = n$. Then $n$ is an integer, $n \le 2014$, and $x = n + \alpha$, where $0 \le \alpha < 1$. Substituting into the "quadratic" and simplifying, we get

$$n\alpha + \alpha^2 = 20{\cdot}15 \,.$$

Now $0 \le \alpha < 1$, so $n$ cannot be negative and we have

$$n + 1 > n\alpha + \alpha^2 \,,$$

so $n \ge 20$. There are $1995$ values of $n$ from $20$ to $2014$; we shall show that for each of these values, there is exactly one possible value of $\alpha$ and hence one possible value of $x$.

Consider $n\alpha + \alpha^2$ as a quadratic in $\alpha$. This quadratic increases continuously from the value $0$ (which obviously is less than $20{\cdot}15$) when $\alpha = 0$ to the value $n + 1$ (which by assumption is greater than $20{\cdot}15$) when $\alpha = 1$; so there is exactly one value of $\alpha$ at which $n\alpha + \alpha^2 = 20{\cdot}15$. Thus we have $1995$ possible value of $n$, each with one associated value of $\alpha$, giving $1995$ values of $x$.

**Q1488** Let $m$ be a integer, $m \ge 2$. Prove that there is a cubic polynomial

$$p(x) = x^3 + ax^2 + bx + c$$

with integer coefficients, such that when $x$ is an integer, $p(x)$ is never a multiple of $m$.

**SOLUTION** Consider the possible remainders when the expression $x^3 - x^2$, for any integer $x$, is divided by $m$. All possible remainders will be found by taking $x = 0, 1, 2, \ldots, m - 1$. This gives $m$ remainders; but they will not all be different, since $x = 0$ and $x = 1$ both give remainder $0$. Therefore there will be an integer $c$ from $0$ to $m - 1$ which is never found as a remainder when $x^3 - x^2$ is divided by $m$; and therefore $p(x) = x^3 - x^2 - c$ never has remainder $0$, that is, it is never a multiple of $m$.

**Q1489** The point $(s,t)$ is the centre of a square. Three vertices of the square lie on the parabola $y = x^2$. If $s = \frac{3}{2}$, find the coordinates of all four vertices of the square.

**SOLUTION** Let the points on the parabola be $A = (a, a^2)$ and $B = (b, b^2)$ and $C = (c, c^2)$, where $\angle BAC$ is a right angle. Then (1) $AB$ and $AC$ are perpendicular, so the product of their gradients is $-1$; (2) $AB$ and $AC$ have equal length; and (3) the point $(s,t)$ is the midpoint of $BC$. Writing these facts as equations and using the given value $s = \frac{3}{2}$, we have

$$\frac{b^2 - a^2}{b - a}\frac{c^2 - a^2}{c - a} = -1 \tag{1}$$

$$(b - a)^2 + (b^2 - a^2)^2 = (c - a)^2 + (c^2 - a^2)^2 \tag{2}$$

$$b + c = 3 \,. \tag{3}$$

Simplifying $(1)$ and factorising $(2)$ gives

$$(b + a)(c + a) = -1 \tag{4}$$

$$(b - a)^2[1 + (b + a)^2] = (c - a)^2[1 + (c + a)^2] \,. \tag{5}$$

Now multiply both sides of $(5)$ by $(b + a)^2$ and substitute from $(4)$ to get

$$(b + a)^2(b - a)^2[1 + (b + a)^2] = (c - a)^2[(b + a)^2 + 1] \,,$$

that is,

$$(b^2 - a^2)^2 = (c - a)^2 \,.$$

There are two possibilities, $b^2 - a^2 = c - a$ or $b^2 - a^2 = a - c$. We shall solve the first and shall leave the reader to check that the second gives the same solution with the points $B$ and $C$ interchanged, which makes no difference in terms of the stated question. So, consider $b^2 - a^2 = c - a$, and use $(3)$ to eliminate $c$ from this equation and from $(4)$. We have

$$b^2 - a^2 = 3 - a - b \tag{6}$$

$$(b + a)(3 + a - b) = -1 \,. \tag{7}$$

Expanding $(7)$ gives $3b + 3a + a^2 - b^2 = -1$; adding $(6)$ to this equation gives $3b + 3a = -1 + 3 - a - b$ and so

$$a + b = \frac{1}{2} \,.$$

Substituting back into $(7)$ leads to

$$a - b = -5$$

and it is now easy to get

$$a = -\frac{9}{4} \,, \quad b = \frac{11}{4} \,, \quad c = \frac{1}{4} \,.$$

Therefore three of the vertices of the square are

$$A = \left(-\frac{9}{4}, \frac{81}{16}\right) , \quad B = \left(\frac{11}{4}, \frac{121}{16}\right) , \quad C = \left(\frac{1}{4}, \frac{1}{16}\right) ,$$

and the fourth, which does not lie on the parabola, is

$$D = B + C - A = \left(\frac{21}{4}, \frac{41}{16}\right) .$$

**Q1490** Let

$$S = \cos 72° + \cos 144° \quad \text{and} \quad T = \cos 72° - \cos 144° .$$

- Prove that $2ST = -T$.
- Hence find the exact value of $\cos 72°$.

**SOLUTION** Write $\alpha = \cos 72°$ and $\beta = \cos 144°$. Using the double angle formula, we have

$$2\alpha^2 - 1 = \cos 144° = \beta \quad \text{and} \quad 2\beta^2 - 1 = \cos 288° = \cos 72° = \alpha .$$

Therefore

$$\begin{aligned} 2ST &= 2(\alpha + \beta)(\alpha - \beta) \\ &= 2(\alpha^2 - \beta^2) \\ &= (2\alpha^2 - 1) - (2\beta^2 - 1) \\ &= \beta - \alpha \\ &= -T . \end{aligned}$$

It is clear that $T \neq 0$, so $S = -\frac{1}{2}$ and the double angle formula from above gives

$$2\alpha^2 - 1 = -\frac{1}{2} - \alpha .$$

Solving the quadratic, and noting that since $72°$ is a first quadrant angle the negative root must be rejected, gives

$$\cos 72° = \alpha = \frac{\sqrt{5} - 1}{4} .$$