Parabola Volume 53, Issue 1 (2017)

Integer Points on Conics and Continued Fractions

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Let us begin with a question:

Question. *Find all the points with integer coordinates on the hyperbola* $x^2 - 8xy + 11y^2 = 1$.

For example, $(x, y) = (25, 4)$ works and so does $(8057, 1292)$.

How do we find all such points? One approach to this is to use continued fractions, so we begin by recapping the basic theory.

Continued Fractions

A *continued fraction* is a way of representing rational and real numbers. We use the notation $[a_0; a_1, a_2, \ldots]$ to mean

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}\,.$$

Thus, the continued fraction $3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{7}}}$ is written as $[3; 1, 5, 7]$. This equals $3\frac{36}{43}$.

Rational numbers have terminating continued fractions, while quadratic irrationals (that is real numbers such as $5 + \sqrt{7}$ which satisfy a quadratic equation with integer coefficients) have eventually recurring continued fractions. Thus, for example,

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \ldots}}}}$$

and is written as $[1; \overline{1, 2}]$.

If N is a non-square positive integer, then the continued fraction for $\sqrt{N}$ has a special form:

$$\sqrt{N} = [a_0; \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}]\,.$$

For example, $\sqrt{54} = [7; \overline{2, 1, 6, 1, 2, 14}]$ and $\sqrt{53} = [7; \overline{3, 1, 1, 3, 14}]$. Such numbers are called *pure quadratic irrationals*.

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By truncating a continued fraction at the nth term, we obtain a rational approximation, $\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$, called a *convergent* to the number represented by the continued fraction. For example, for the continued fraction $\sqrt{3} = [1; 1, 2, 1, 2, 1, 2, \ldots]$,

$$\frac{p_0}{q_0} = 1 = \frac{1}{1}\,, \qquad \frac{p_1}{q_1} = 1 + \frac{1}{1} = \frac{2}{1}\,, \qquad \frac{p_2}{q_2} = 1 + \frac{1}{1+\frac{1}{2}} = \frac{5}{3}\,,$$

and so on. There is a simple recurrence formula for finding p_n and q_n, namely

$$\begin{aligned} p_{n+1} &= a_{n+1}p_n + p_{n-1} \\ q_{n+1} &= a_{n+1}q_n + q_{n-1}\,. \end{aligned}$$

Pell's Equation

It is well-known that certain of the convergents of the continued fraction for $\sqrt{N}$, where N is a positive non-square integer, provide solutions to the Pell equation $x^2 - Ny^2 = 1$. Here, for example, is a table showing the first few convergents for $\sqrt{3} = [1; \overline{1, 2}]$.

n	0	1	2	3	4	5	6	7
a_n	1	1	2	1	2	1	2	1
p_n	1	**2**	5	**7**	19	**26**	71	**97**
q_n	1	**1**	3	**4**	11	**15**	41	**56**
$p_n^2 - 3q_n^2$		1		1		1		1

and for n odd, $p_n^2 - 3q_n^2 = 1$.

Non-pure quadratic irrationals

One can consider the convergents also for non-pure quadratic irrationals such as $2+\sqrt{3}$ and ask whether the corresponding numerators P_n and denominators Q_n satisfy some related *Pell-type* equation. For example, the table showing the partial quotients for $2 + \sqrt{3} = [3; \overline{1, 2}]$ begins as follows:

n	0	1	2	3	4	5	6	7
a_n	3	1	2	1	2	1	2	1
P_n	3	**4**	11	**15**	49	**56**	153	**209**
Q_n	1	**1**	3	**4**	11	**15**	41	**56**

Taking the odd-numbered terms, what Pell-type equation do the pairs

$$(P_n, Q_n) = (4, 1), (15, 4), (56, 15), (209, 56), \ldots$$

satisfy?

Matrix Representation for Convergents

If p_n, q_n are defined by

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

then

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, a_3, \ldots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} .$$

The proof of this is an easy induction and can be found in [1].

Theorem. *Let N be a positive non-square integer. If $\frac{p_n}{q_n}$ is a convergent for $\sqrt{N}$ such that $p_n^2 - Nq_n^2 = 1$ and if $\frac{P_n}{Q_n}$ is the convergent of $a + \sqrt{N}$, then*

$$(P_n - aQ_n)^2 - NQ_n^2 = 1 .$$

Proof. Write $\sqrt{N} = [b; a_1, a_2, \ldots]$.
Then, for some n, there is a partial quotient $\frac{p_n}{q_n}$ for $\sqrt{N}$ such that $p_n^2 - Nq_n^2 = 1$, where

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

Now, $a + \sqrt{N} = [a + b; a_1, a_2, \ldots]$, so

$$\begin{aligned} \begin{pmatrix} P_n \\ Q_n \end{pmatrix} &= \begin{pmatrix} a+b & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a+b & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a+b & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -b \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_n + aq_n \\ q_n \end{pmatrix} . \end{aligned}$$

Hence, $P_n = p_n + aq_n$ and $Q_n = q_n$, and so

$$1 = p_n^2 - Nq_n^2 = (P_n - aQ_n)^2 - NQ_n^2 . \qquad \square$$

Thus, in the example above, $2 + \sqrt{3} = [3; \overline{1, 2}]$ begins as follows:

n	0	1	2	3	4	5	6	7
a_n	3	1	2	1	2	1	2	1
P_n	3	**4**	11	**15**	49	**56**	153	**209**
Q_n	1	**1**	3	**4**	11	**15**	41	**56**
$(P_n - 2Q_n)^2 - 3Q_n^2$		1		1		1		1

since

$$\begin{aligned} (4 - 2 \times 1)^2 - 3 \times 1^2 &= 1 \\ (15 - 2 \times 4)^2 - 3 \times 4^2 &= 1 \\ (56 - 2 \times 15)^2 - 3 \times 15^2 &= 1 , \end{aligned}$$

and so on.

The above proof shows that if $\frac{p_n}{q_n}$ is a convergent for $\sqrt{N}$ such that $p_n^2 - Nq_n^2 = T$, then $(P_n - aQ_n)^2 - NQ_n^2 = T$, where $\frac{P_n}{Q_n}$ is the convergent of $a + \sqrt{N}$.

Back to the start

We return to the original question:

Question. *Find all the points with integer coordinates on the hyperbola* $x^2 - 8xy + 11y^2 = 1$.

We can complete the square and write $(x - 4y)^2 - 5y^2 = 1$ and, hence, we look at the convergents $\frac{P_n}{Q_n}$ of $4 + \sqrt{5}$ which has continued fraction $[6; \overline{4}]$.

n	0	1	2	3	4	5
a_n	6	4	4	4	4	4
P_n	6	**25**	106	**449**	1902	**8057**
Q_n	1	**4**	17	**72**	305	**1292**
$P_n^2 - 8P_nQ_n + 11Q_n^2$		1		1		1

Hence, every second convergent in the table above will produce a point on the hyperbola with positive integer coefficients, and conversely every point on the hyperbola with positive integer coefficients will be one of the convergents in the table above.

References

[1] R.F.C. Walters, *Number Theory: An Introduction*, Carslaw Publications, Sydney, 1986.