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Charting new territory: Formulating the Dalivian coordinate system

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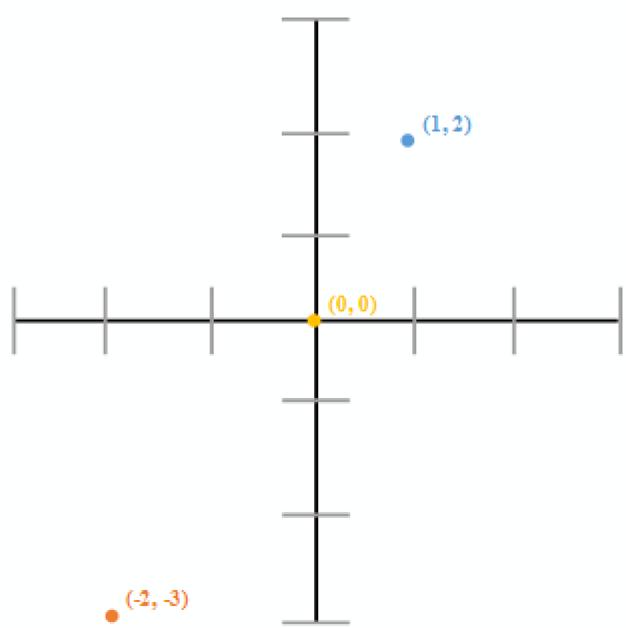
Numerous coordinate systems have been invented. The very first and most widely known coordinate system is the Cartesian system. In this system, points are located by their relative distance from perpendicular intersecting lines, called *axes*. The horizontal axis is called the x*-axis* and the vertical axis is called the y*-axis*. All points, lines, and figures are drawn in the coordinate plane formed from the intersection of the two axes. The point where the two axes intersect is called the *origin* and is the starting point for plotting points. The x-axis is positive in the area to the right of the origin and negative in the area to the left of the origin, while the y-axis is positive in the space above the origin and negative in the space below the origin. Points are plotted as (x, y). Equations can be used to create ellipses, circles, and curved lines. Below are various points plotted using the Cartesian system:



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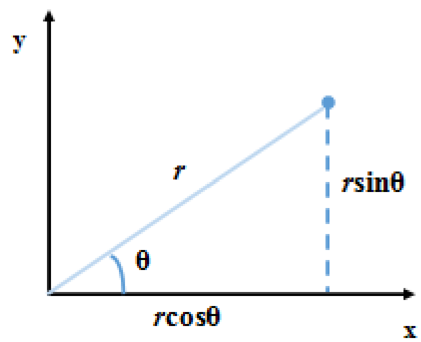
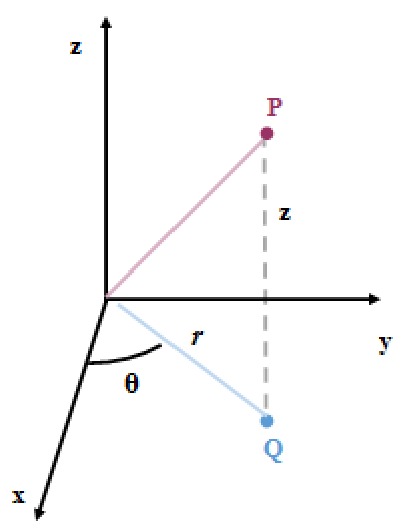
The polar coordinate system is perhaps the most widely known system other than the Cartesian system. The polar coordinate system is used to plot a complex curve, known as a spiral. The polar system plots points using coordinates involving an origin point and axis lines. Every point in the system is described in terms of the directed distance from the point to the origin, and the rotation the directed distance line makes with the polar axis. A point is represented by the coordinate (r, θ). Here, θ is the angle in radians from the polar axis and r is the distance from the pole along the directed distance line. When the angle of rotation is positive, the directed distance goes in a counterclockwise direction; when it is negative, the directed distance moves in a clockwise direction [1]. Shown below is the relationship between r and θ.



Coordinate systems can also be used for mapping. Cylindrical coordinates are one example of this. Cylindrical coordinates are polar coordinates represented in three dimensions. Cylindrical coordinates use traditional polar coordinates and add a third dimension, z, which is the distance from the point P in space along the z-axis to the xy-plane. This is the same z as the z used to graph three-dimensional coordinates in Cartesian space. Polar coordinates (r, θ, z) are shown below for the point P [2].



The Dalivian coordinate system

There are many coordinate systems that differ greatly from Cartesian, polar, and cylindrical coordinate systems. In fact, there is only one requirement for a coordinate system to exist; conversion formulas must allow the conversion from one system to another and vice versa. The focus of our research is to develop a new way of graphing coordinates in a coordinate plane. Our system, the Dalivian system, builds on ideas from the Cartesian system. The name "Dalivian" is derived from the names of its creators. We will create a system based on the intersection of two parabolas using the equations $x = ay^2$ and $y = bx^2$. The point (x, y) is the point of intersection between the two parabolas in the Cartesian system. However, the point will be labeled $(a|b)$ in the Dalivian system. Here, a represents parabolas opening to the left and right while b represents parabolas opening up and down.

The origin is always a point of intersection between the two parabolas. For this reason, the point (x, y) will always be the non-origin intersection of the two parabolas and $(0, 0)$ will never be displayed on the graph. One of our focal points and major issues will be to find conversion formulas between the Dalivian system and the Cartesian coordinate system. These would allow us to easily convert to and from the Cartesian system. We will also focus on graphing equations in our system. We do not know what the graphs of equations will look like in the Dalivian system but we believe these graphs will not resemble anything we have studied before.

Abstract

The Dalivian coordinate system presents a new way to graph points in a coordinate plane. The traditional way to graph points, in the Cartesian plane, uses an x-axis and a y-axis to plot points horizontally and vertically in a grid. The Dalivian system differs from the traditional system in that points are graphed using the non-origin intersection of two parabolas, $x = ay^2$ and $y = bx^2$. Points are labeled $(a|b)$, with a representing parabolas opening left and right and b representing parabolas opening up and down. Our research mainly focused on identifying generalizations that can be made in the Dalivian system, which later transitioned to proving a few of these generalizations. An integral part of the research was developing conversion formulas. These allow for a conversion between points in the Dalivian and Cartesian systems. They are

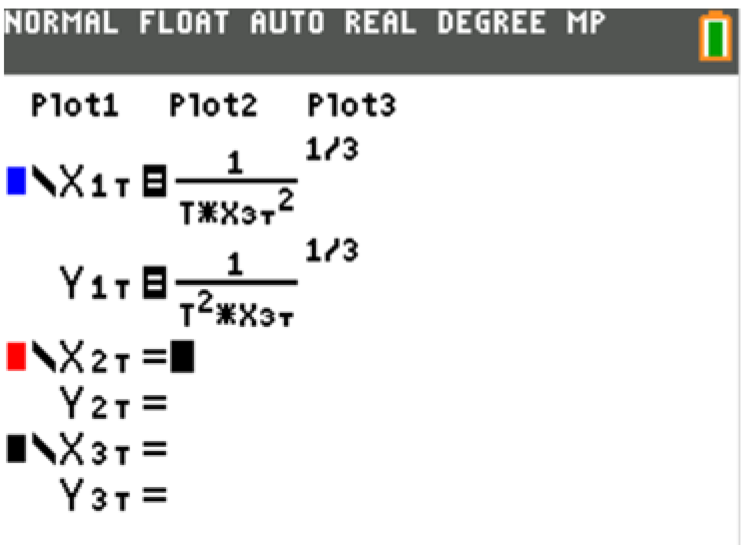
$$\begin{aligned} x &= \sqrt[3]{\frac{1}{ab^2}} \qquad & a &= \frac{x}{y^2} \\ y &= \sqrt[3]{\frac{1}{a^2b}} \qquad & b &= \frac{y}{x^2}\,. \end{aligned}$$

Another major part of the research was creating graph paper so one could graph points, labeled as $(a|b)$, without having to convert to x and y coordinates. The graph paper in the Dalivian system involves four sets of ten parabolas, traditionally starting at $y = x^2$, and becoming increasingly narrower. There are a total of forty parabolas featured on the graph, with ten opening in each of the four directions.

Methodology and Results

We began our research by attempting to find conversion formulas. These would allow us to convert points from the Dalivian System to the Cartesian System, and vice versa. We found our conversion formulas from the Dalivian system to the Cartesian system through substitutions in the equations $x = ay^2$ and $y = bx^2$. It was found that the formulas were $x = \sqrt[3]{\frac{1}{ab^2}}$ and $y = \sqrt[3]{\frac{1}{a^2b}}$. By solving for a and b in $x = ay^2$ and $y = bx^2$, we developed our conversion formulas from the Cartesian system to the Dalivian system, $a = \frac{x}{y^2}$ and $b = \frac{y}{x^2}$. Having conversion formulas gave us an efficient way to graph more complex equations.

Once we had the conversion formulas, we decided that instead of graphing each equation by hand, we use parametric equations and our calculators. This is shown below:



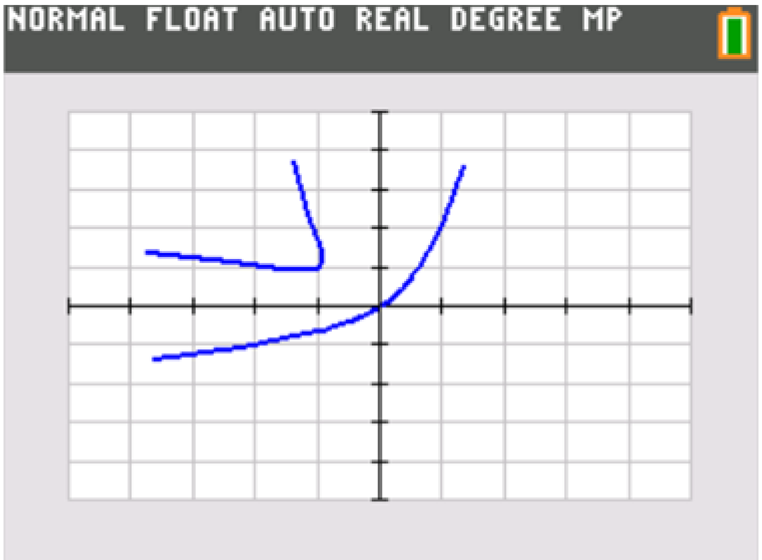
Here, T represents a and x_{3T} represents b in the Dalivian system. Once the conversion formulas are entered, one is able to input any Dalivian equation that is solved for b in x_{3T} and its graph will be displayed. This provided us with a simple and quick way to make generalizations, as we could instantly view any equation's graph.

We began by looking at the effects of negatives on graphs. What we found is that when the sign of a is flipped, the graph reflects over the y-axis. When a negative is applied to a whole equation, the graph reflects over the x-axis in the Dalivian system.

We confirmed that our quadrants are the same quadrants as in the Cartesian system. For instance, when a and b are positive, a point is graphed in the upper right quadrant, similar to when x and y are positive. A difference between the two systems is that no point ever will be graphed on either axis in the Dalivian system. This is because the intersection of two parabolas with vertices at the origin cannot cross an axis as neither of the parabolas can touch an axis. The origin is a point of discontinuity in the Dalivian system. When zero is plugged in for x or y in $x = ay^2$ and $y = bx^2$, there is no specific value for a or b. In addition, when x, y, a, or b is equal to zero, our conversion formulas produce a dividing by zero error.

A difference between graphing in the Dalivian system and graphing in the Cartesian system is the way that numbers are labeled. Our numbers are labeled starting with values farthest from zero, as opposed to numbers closest to zero as in the Cartesian system. The Dalivian system's parabolas are labeled this way because a and b values closer to zero correspond to wider parabolas, so they will be farther from the axes than more extreme a and b values with skinnier parabolas that stay closer to the axes.

We also examined the equation $b = a + c$. The general shape of this graph is shown below:

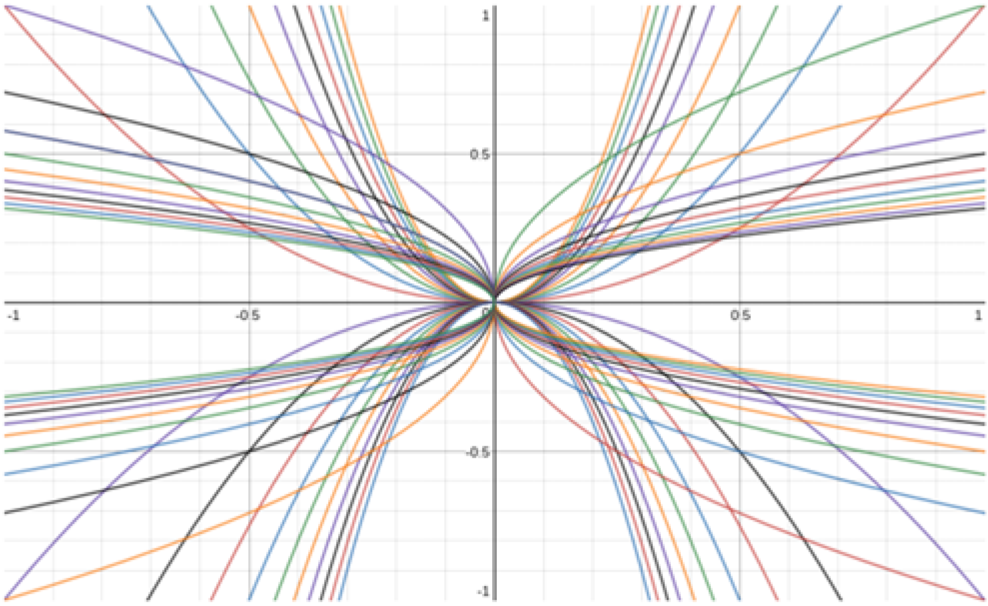


There are two shapesbeing made from this equation. When c increases, the shapes move closer together. We found that the distance from the shape in the upper left quadrant to the origin is $\sqrt{8/c^2 2}$.

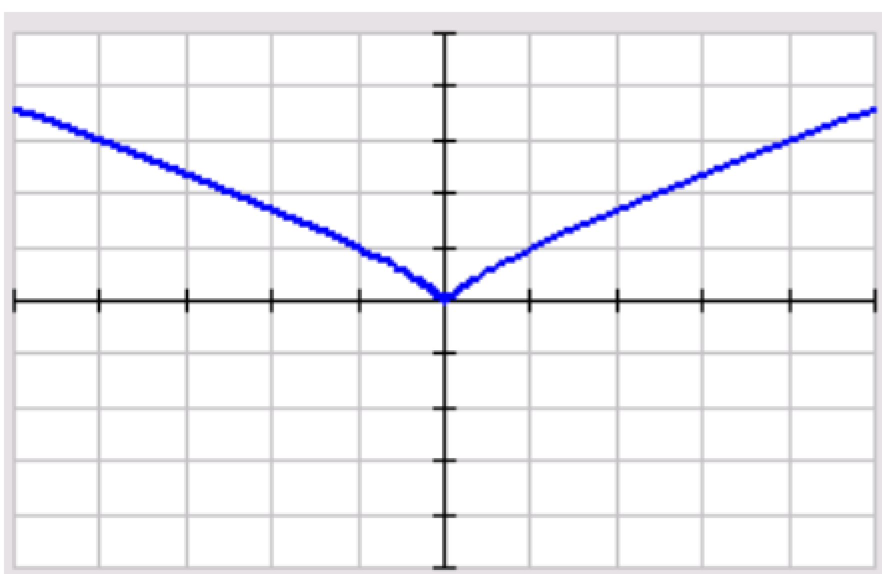
We then began creating graph paper for our system. We used Desmos [3], an online graphing calculator, to make forty parabolas. All of these have their vertex at the origin, with ten parabolas opening in each direction. For our original graph paper, we used the equations $x = ay^2$ and $y = bx^2$. We started with a and b equal to 1, and then changed the a and b values of each equation by one integer, stopping once a and b were equal to 10 and -10 in the skinniest parabolas. We then experimented with differently scaled graphs that allowed us to graph equations that either did not produce integers for b when an integer was plugged in for a, or were outside of the range of our graph. For instance, the equation $b = a^2 + a$ only shows up in the upper right and upper left quadrants when graphed with integers, or in the upper right and lower left quadrants when graphed with a between .1 and 1, but appears in all three quadrants when graphed with fractions from $(.1, 1)$ as well as integers from $(1, 10)$.

Our newly formed graph paper shows the intersection points of parabolas, which is what the Dalivian system is based on. Using the graph paper to show an equation is quite simple. Plug in a number for a, then find the corresponding b value. After that, find the parabola corresponding to the a value and the parabola corresponding to the b value. Their intersection forms a point. Shown below is the graph paper where a and b

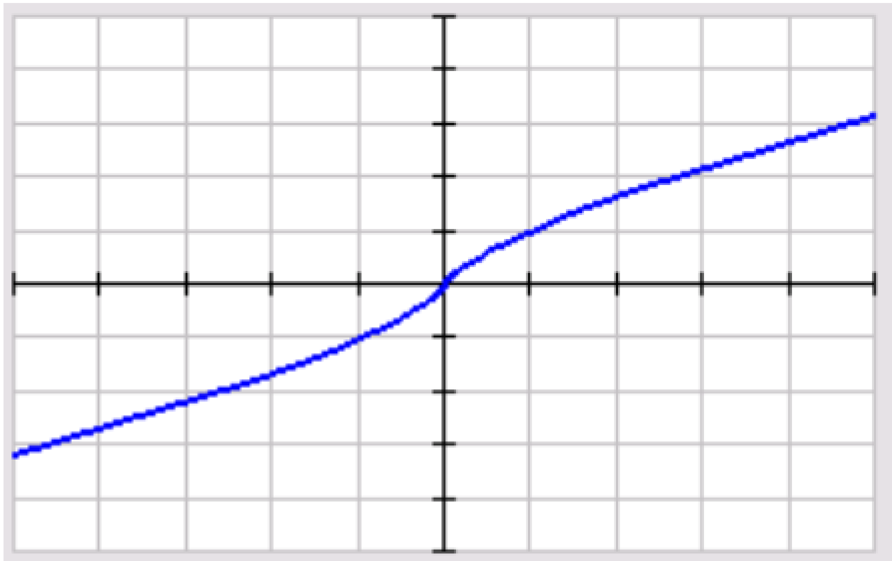
are integers from $(-10, 10)$ and the window ranges from $(-1, 1)$ for both the x-axis and the y-axis:



Having graph paper made it easier to see the symmetry present in each equation, which is what we looked at next. We had a general idea of which equations were symmetric and which ones were not, but we did not know definitively. After filtering through the graphs we knew were unlikely to be symmetric, we began to use algebra to prove which ones that showed signs of symmetry were actually symmetric. We found that $b = a^2$, $b = a^3$, $b = \frac{1}{a}$, and $b = ka$ were all symmetric. The equation $b = a^2$ was a reflection over the y-axis, while the other three were 180 degree rotations.



Shown above is a^2



Shown above is a^3

One of our last steps in our research was to create proofs. We proved that the quadrant rules are the same in the Cartesian and Dalivian systems both in words as well as in symbols. To prove in symbols, we plugged positive and negative x and y values into our conversion formulas to determine if the corresponding a or b value would also be positive or negative. x corresponds to a and y corresponds to b, as a represents numbers in the horizontal direction and b represents values in the vertical direction. For instance, when a positive a and negative b were inputted into our conversion formulas, the result was a positive x and negative y.

We also proved the behavior of graphs at intercepts. The Dalivian system does not have traditional intercepts since points can never be graphed on axes. Instead, the graph moves from one quadrant to the next when a or b passes through zero. The numbers a and b cannot be zero because no parabola has a width of zero. A graph jumps quadrants instead of just producing a hole because, when a or b passes through zero, a or b changes signs. This means that our graphs do not have a range of infinity, since they only extend as far as the point closest to $a = 0$ or $b = 0$ before switching quadrants.

Discussion and Conclusions

The Dalivian system is an example of mathematical theory. Since a theory is considered to be a body of knowledge, mathematical theory can be considered an area of mathematical research. For that reason, the research that we undertook can be labeled as a mathematical theory. The benefits of math theory include its enjoyability, challenge, and satisfaction.

One of the areas of research that we failed to complete was defining a function. We determined that a function can be defined the same way as in the Cartesian system; each input has one output. However, the test for a function cannot be the vertical line test. Since a represents a sideways parabola, there can be many points that fall in a vertical line but are on different parabolas and therefore have different a values. In our system, one output for each input means that each a value can only have one b value. This means that each sideways parabola can only have one intersection point. We could not determine an effective way to test for a function in the Dalivian system, so future research could focus on developing an efficient way to test for functions.

References

[1] A.V. Johnson, *Coordinate System, Polar.*, retrieved from http://www.encyclopedia.com/doc/1G2-3407500077.html on 2017-10-25.

[2] D.Q. Nykamp, *Cylindrical Coordinates. Math Insight*, retrieved from http://mathinsight.org/cylindrical_coordinates on 2017-10-25.

[3] *Desmos*, retrieved from https://www.desmos.com/ on 2017-10-25.