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Proof of the Irrationality of the Square Root of 2 Contained in Babylonian Geometry Problem Tablets

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Abstract

We show that Old Babylonian problem tablets contain a geometric proof of the irrationality of $\sqrt{2}$ predating the Greek discovery of this profound mathematical fact by more than a millennium.

One of the greatest achievements of Ancient Greek mathematics is the proof that $\sqrt{2}$ is irrational [1, p. 48]. The proof is traditionally credited to the circle of Pythagoras (c. 570- c. 495 BCE) [2]; however, this specific attribution is disputed [1, p. 148]. The Greek proof is algebraic and proceeds by contradiction: assume that $\sqrt{2}$ is rational so therefore it can be written as the quotient of two integers $\frac{p}{q}$ with this fraction in lowest terms. Simple algebraic manipulations then quickly yield a contradiction as both p and q are found to be even so the quotient is not in lowest terms. A century after Pythagoras Theodorus of Cyrene (5th century BCE) was able to prove the irrationality of $\sqrt{3}$, $\sqrt{5}$ and other numbers up to $\sqrt{17}$ [3, pp. 396–404].

Figure 1 shows an Old Babylonian tablet known as BM15285 (c. 1800–1600 BCE) [4,5]. The text on accompanying the relevant figure on the tablet states:

"The side of the square is 60 rods. Inside it [I drew] 16 wedges [triangles]. What are their areas?"

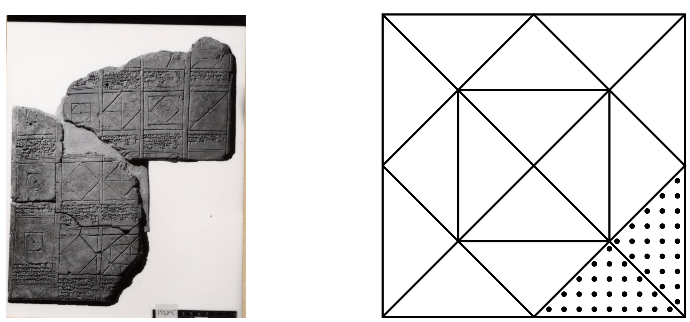

Figure 1: BM15285 and the relevant detail

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Figure 1 is Problem 12 of a presumed 41 geometric area problems. The speckled region indicates a portion of the tablet that is broken off.

Remarkably, this Old Babylonian tablet problem contains a geometric version of the Greek proof of the irrationality of 2. To show this, we first show that the tablet figure contains a geometric demonstration of the *Pythagorean Theorem* for the special case of an isosceles right triangle. We begin by showing that, by using only a compass and straight edge, Figure 1 can be drawn so that the sixteen triangles have the same area. See Figure 2 where the vertices on Figure 1 have been marked.

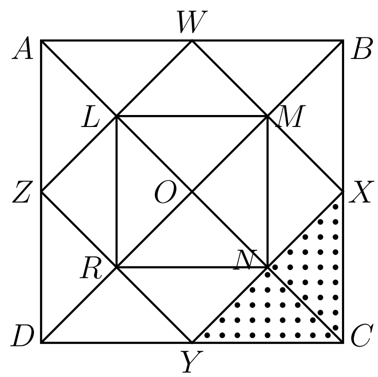

Figure 2

Draw the large square $ABCD$ by constructing lines perpendicular to one edge and marking off on these lines segments of length of the first edge. Draw the two diagonals and call the midpoint (by symmetry) O. On segments AO, BO, CO and DO, find midpoints L, M, N, and R. Draw lines through L, M, N, and R perpendicular to AC, BD, AC and BD, respectively, and call the intersection of these lines with the sides of the large square W, X, Y and Z. Finally, connect points L, M, N, and R. By symmetry, all sixteen triangles have equal areas that we denote as T.

Consider now the isosceles right triangle (by the above construction) LMN. By considering square $LMNR$, we see that $(LM)^2 = (MN)^2 = 4T$. By considering square $WXYZ$, we see that $(LN)^2 = 8T$. Thus, $(LM)^2 + (MN)^2 = (LN)^2$ and the Pythagorean Theorem for isosceles triangles is established from the figure. Another way to write this, if we denote by H the length of the hypotenuse of an isosceles right triangle and S by the length of a side, is $H^2 = 2S^2$. From this, we also see immediately that the hypotenuse of an isosceles right triangle must be longer than a side.

Now, consider the large square $ABCD$ and assume that $\sqrt{2}$ is rational. Then, denoting by S the length of the side of the large square AB and H the length of the main diagonal AC, we have since $H^2 = 2S^2$ that $\frac{H}{S} = \sqrt{2}$ is a rational number which has been reduced to lowest terms. But consider triangle AOB, itself by construction also an isosceles right triangle. The hypotenuse of AOB is AB with length S. The sides of AO and BO are by symmetry of length $\frac{H}{2}$. Since H^2 is equal to twice the integer S^2, H^2 is even. But if the integer H^2 is even, H also must be even. Thus, $\frac{H}{2}$ is an integer. By the Pythagorean Theorem proved above, $\frac{S}{\frac{H}{2}} = \sqrt{2}$. But since S and $\frac{H}{2}$ are integers and $\frac{H}{2} < S < H$, this contradicts that $\frac{H}{S}$ is reduced to lowest terms. Thus, $\sqrt{2}$ is irrational.

Problem 12 from BM15285 appears to be the last in a series of problems emphasizing that the area of a square inscribed in another square is half of the area of the outer square, and the possibility of repeated such nesting. Problem 7 [4, 5] states:

"The side of the square is one [unit]. Inside it I drew a second square. The square that I drew touches the outer square. What is its area?"

Problems 8, 10 and 11 are very similar to Problem 7. Problem 9 [4, 5], which we leave to the reader as an exercise to draw the relevant figure and solve, states:

"The side of the square is one [unit]. In it I drew a square.
The square that I drew touches the [first] square.
Inside the second square I drew a third square.
[This square] that I drew touches the [second] square.
What is its area?"

Interestingly, while another Old Babylonian tablet known as YBC 7289 [6] (Figure 3) contains the best three digit sexagesimal (base 60) approximations, accurate to six decimal digits, of both $\sqrt{2}$ and its reciprocal, it has not been thought previously by mathematicians [7] or historians of mathematics [1, p. 48], [3, pp. 405–409], [8] that the Babylonians knew, or certainly did not prove, that $\sqrt{2}$ is irrational. This tablet, like BM 15285, while avoiding working explicitly with $\sqrt{2}$, cleverly works around it. Indeed, we can now see that, in addition to the numeric approximation, YBC 7289 also contains the same irrationality proof as BM 15285, if one assumes the Pythagorean Theorem equating the sum of the squares of the lengths of the legs of an isosceles right triangle to the square of the length of its hypotenuse. Perhaps presciently, Neugebauer noted [1, p. 48]:

"But all the foundations were laid which could have given this result [irrationality of $\sqrt{2}$] to a Babylonian mathematician [...]"

As we now see, Babylonian geometric problem tablets indeed contain a proof of this fact! Further investigation is merited to see if indeed the Babylonians knew and appreciated the implications of the proof contained in their tablets and thus that the Babylonian geometers predated the Greeks by a millennium in proving this profound fact. Regardless, the Babylonian problem figure is a "Book Proof" — indeed, a "Tablet Proof", of the irrationality of the square root of 2.

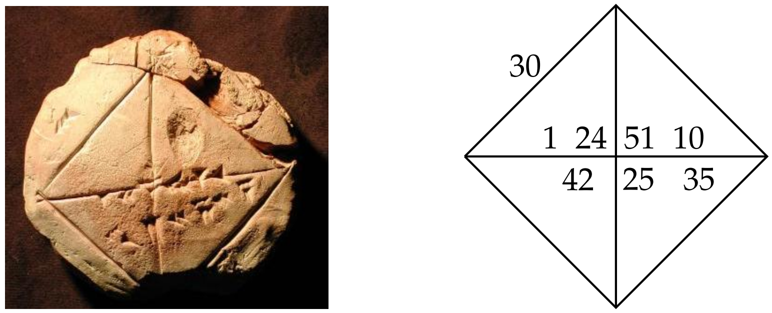

Figure 3: Rendering of YBC 7289 with Arabic numbers.
The side length is 30 sexagesimal, or $\frac{30}{60} = \frac{1}{2}$ decimal. The diagonal length is 42 25 35 sexagesimal, or $\frac{42}{60} + \frac{25}{3600} + \frac{35}{216000} = 0.70710648148$ ($\sqrt{2}/2 \approx 0.7071067$). Also, 1 24 51 10 sexagesimal, or $1 + \frac{24}{60} + \frac{51}{3600} + \frac{10}{216000} = 1.41421296296$, represents the reciprocal of the diagonal length $2/\sqrt{2} = \sqrt{2} \approx 1.414213$.

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