Parabola Volume 54, Issue 1 (2018)

Solutions 1541–1550

Q1541 Consider the equation

$$29x + 30y + 31z = 366 \,, \qquad (*)$$

where x, y, z are positive integers with $x < y < z$.

(a) Without any writing or computer assistance, find x, y, z which satisfy these conditions.

(b) Prove that your solution from (a) is the only possibility.

SOLUTION It is well known that a leap year contains 1 month of 29 days, 4 months of 30 days and 7 months of 31 days, a total of 366 days. That is,

$$(29 \times 1) + (30 \times 4) + (31 \times 7) = 366 \,,$$

and so a solution is $x = 1$, $y = 4$, $z = 7$. To show that there is no other solution, we consider two possibilities: $x = 1$ or $x \geq 2$.

If $x = 1$, then we have

$$30y + 31z = 337 \,.$$

Since we know one solution $y = 4$, $z = 7$, we can use the technique explained in *Parabola* Vol. 49, No. 2 (2013) to find all possible solutions

$$y = 4 - 31t \,, \quad z = 7 + 30t$$

where t is an integer. But if $t > 0$, then y is negative, which is not allowed; if $t < 0$, then z is negative, which is not allowed; and if $t = 0$, then the solution is the one we have already found.

Finally, consider the case $x \geq 2$. We have $y > x$ and so $y \geq 3$. Also, our original equation can be written as

$$z - x = 6(61 + 5x - 5y - 5z) \,;$$

so $z - x$ is a multiple of 6; and therefore $z \geq 8$. Substituting back into $(*)$ we have

$$\text{LHS} \geq (29 \times 2) + (30 \times 3) + (31 \times 8) = 396 \,,$$

and so $29x + 30y + 31z$ cannot equal 366. Thus the solution $x = 1$, $y = 4$, $z = 7$ found in (a) is the only possibility.

Q1542 Solve the equation

$$x = \sqrt{2017 + \sqrt{2017 + x}}\,.$$

SOLUTION Repeatedly substituting the expression on the right hand side back into itself, we have

$$\begin{aligned} x &= \sqrt{2017 + \sqrt{2017 + \sqrt{2017 + \sqrt{2017 + x}}}} \\ &= \sqrt{2017 + \sqrt{2017 + \sqrt{2017 + \sqrt{2017 + \cdots}}}}\,. \end{aligned}$$

If we look carefully at this we will see that the expression under the first square root sign is 2017 plus the whole expression. Thus

$$x = \sqrt{2017 + x}\,.$$

Squaring both sides leads to the quadratic equation

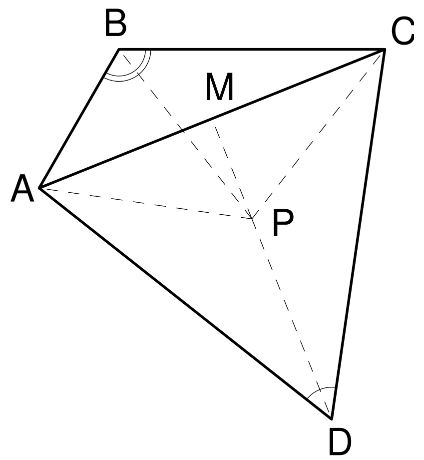
$$x^2 - x - 2017 = 0\,;$$

since x is positive, the solution is

$$x = \frac{1 + \sqrt{8069}}{2}\,.$$

Q1543 Let ABC be a triangle with $\angle ABC = 120°$, and let P be the circumcentre of the triangle, that is, the point for which the lengths AP, BP and CP are all equal. Prove that the ratio of lengths AC/AP is equal to $\sqrt{3}$.

SOLUTION Draw an equilateral triangle ACD on the side AC of the given triangle. Now $\angle ABC = 120°$ and $\angle ADC = 60°$; so opposite



angles of the quadrilateral $ABCD$ add up to $180°$, and it is a cyclic quadrilateral. Hence there is a point P, the centre of the circumscribed circle, which is equidistant from all of A, B, C and D. Since P is equidistant from A, B, C it is the point we require; and since it is equidistant from A, C, D it is not hard to calculate AP. We have $\angle PAC = 30°$, and so

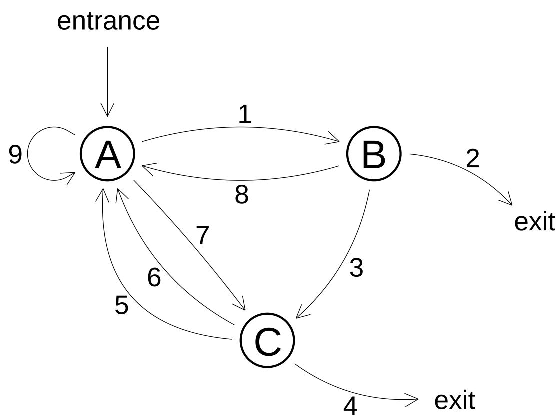
$$\frac{1}{2} AC = AM = AP \cos 30° \ ,$$

that is,

$$\frac{AC}{AP} = 2 \cos 30° = \sqrt{3}$$

as claimed.

Q1544 To reach an exit of the MessConnex tollway, drivers have to get through a system of three roundabouts, shown as A, B and C in the diagram.



Each roundabout has three outgoing roads; two of the nine roads lead out of the MessConnex; the others stay within the system, one of them even returning to the very same roundabout. There are no signs to indicate the correct exit, so the drivers just have to guess; and all the exits look identical, so if drivers return to the same roundabout, they just have to guess again.

The numbers shown on the outgoing roads are the toll (in dollars) charged for using each road. Clearly, a lucky driver could get out of the system for \$3; but most drivers would have to spend much more. How much would it cost the average driver?

SOLUTION Let \$$a$, \$$b$ and \$$c$ be the average amount required before exiting if the driver starts at roundabout A, B or C respectively. Starting from A, the driver has a $\frac{1}{3}$ chance of taking the road to B; in this case they will pay \$1 for taking that road, plus, on average, another \$$b$ to get out. There is also a $\frac{1}{3}$ chance of looping back to A; this costs \$9 plus, on average, another \$$a$. Finally, there is a $\frac{1}{3}$ chance of going to C, which costs \$7 immediately plus, on average, another \$$c$. The total average cost is

$$a = \frac{1}{3}(1 + b) + \frac{1}{3}(9 + a) + \frac{1}{3}(7 + c) \ .$$

Applying similar ideas to drivers who start from B or from C gives the equations

$$b = \frac{1}{3}(8+a) + \frac{1}{3}(2) + \frac{1}{3}(3+c)$$

and

$$c = \frac{1}{3}(5+a) + \frac{1}{3}(6+a) + \frac{1}{3}(4)\ .$$

Note that the last term in the last equation corresponds to choosing the exit from C, in which case the driver pays the $4 toll but has nothing further to pay. Simplifying these equations yields

$$\begin{aligned} 2a - \ \ b - \ \ c &= 17 \\ -a + 3b - \ \ c &= 13 \\ -2a \qquad + 3c &= 15 \end{aligned}$$

which can be solved (exercise!) to give $a = 36$, $b = 26$, $c = 29$. Since the first roundabout the driver reaches is A, the average cost to get out of MessConnex is $36.

Q1545 The numbers $1, 2, 3, \ldots, mn$ are arranged in an array of m rows and n columns in such a way that each of the m rows has the same sum, and each of the n columns has the same sum. Prove that m and m are either both even, or both odd.

SOLUTION The total of all the numbers is

$$1 + 2 + 3 + \cdots + mn = \frac{mn(mn+1)}{2}\ .$$

As this total is distributed equally over m rows, it must be a multiple of m; and as it is distributed equally over n columns, it must be a multiple of n. Therefore

$$2m \mid mn(mn+1) \quad \text{and} \quad 2n \mid mn(mn+1)\ ,$$

where the notation $a \mid b$ means that b is a multiple of a. This implies that

$$n(mn+1) \quad \text{and} \quad m(mn+1)$$

are both even. If either m or n is odd, then $mn+1$ must be even, so mn is odd, so both m and n are odd. That is, m and m are either both even or both odd, as required.

Q1546 (a) Let n, a, b be positive integers such that

$$n^2 < a < b < (n+1)^2\ .$$

Prove that ab cannot be a perfect square.

(b) Find infinitely many examples of positive integers n, a, b, c such that

$$n^3 < a < b < c < (n+1)^3$$

and abc is a perfect cube.

SOLUTION

(a) Taking (positive) square roots of the given inequality, we have

$$n < \sqrt{a} < \sqrt{b} < n + 1\ ,$$

and hence

$$0 < \sqrt{b} - \sqrt{a} < 1\ .$$

Squaring both sides,

$$0 < a + b - 2\sqrt{ab} < 1\ ,$$

and rearranging this inequality yields

$$a + b - 1 < 2\sqrt{ab} < a + b\ .$$

But $a+b-1$ and $a+b$ are consecutive integers; so $2\sqrt{ab}$, which lies between them, cannot be an integer; so $\sqrt{ab}$ cannot be an integer; that is, ab cannot be a perfect square.

(b) If $n = 2$, then

$$(n + 1)^{3/2} - n^{3/2} = \sqrt{27} - \sqrt{8} > \sqrt{25} - \sqrt{9} = 2\ ;$$

since the graph of $y = x^{3/2}$ is concave upwards, values of the left hand side for $n > 2$ are even larger than this. That is, for any integer $n \geq 2$ we have $(n + 1)^{3/2} - n^{3/2} > 2$; so there are integers p, q such that

$$n^{3/2} < p < q < (n + 1)^{3/2}\ ;$$

consequently

$$n^3 < p^2 < pq < q^2 < (n + 1)^3\ .$$

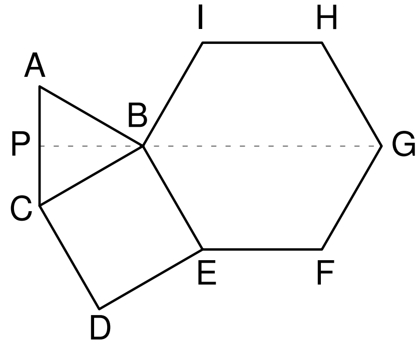
So the integers $a = p^2$, $b = pq$, $c = q^2$ satisfy the required inequality, and their product is

$$abc = (p^2)(pq)(q^2) = (pq)^3\ ,$$

a cube.

Q1547 If the points $A, B, C, D, E, F, G, H, I$ all lie in a plane, and ABC is an equilateral triangle with side length 1, and $BCDE$ is a square, and $BEFGHI$ is a regular hexagon, find the distance AG.

SOLUTION Draw the line BG, meeting AC at P. Now $\angle GBE = 60°$ and $\angle CBE = 90°$, so $\angle CBP = 30°$ and BP is the perpendicular



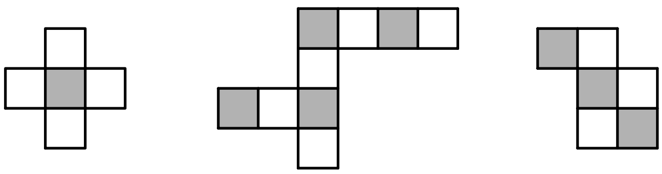
bisector of AC. So $\angle APG$ is a right angle; we have

$$AG^2 = AP^2 + PG^2 = \Big(\frac{1}{2}\Big)^2 + \Big(2 + \frac{\sqrt{3}}{2}\Big)^2 = 5 + 2\sqrt{3}$$

and so

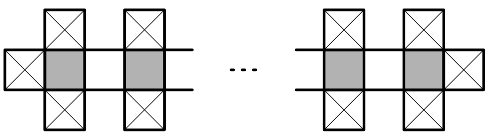
$$AG = \sqrt{5 + 2\sqrt{3}}\ .$$

Q1548 A *polyomino* is a figure consisting of unit squares joined along their edges. Every join must involve the full edge of both squares. We can give a polyomino a "chessboard" colouring (alternately light and dark) and calculate the ratio of light to dark squares. The diagram shows some possibilities with ratios $\frac{4}{1} = 4$, $\frac{5}{4}$ and $\frac{3}{3} = 1$.



Prove that if $\frac{m}{n}$ is a rational number between $\frac{1}{3}$ and 3, then there is a chessboard–coloured polyomino such that the ratio of light to dark squares equals $\frac{m}{n}$.

SOLUTION First suppose that $1 \le \frac{m}{n} \le 3$. Consider the following polyomino which contains n dark squares, $n - 1$ light squares, and $2n + 2$ further squares (marked with crosses) which are optional, and will be light if they are used.



Since $n \le m \le 3n$, we have $1 \le m - n + 1 \le 2n + 1$; so we can choose $m - n + 1$ of the "optional" squares to be light, and omit the rest. Then the total number of light squares is m, and the ratio is $\frac{m}{n}$ as required.

In the case $\frac{1}{3} \le \frac{m}{n} \le 1$, we can obtain a ratio of $\frac{n}{m}$ as above; then interchange light and dark squares.

Q1549 Let a, b be positive integers with no common factor, and let n be an integer, $n \geq 2$. Prove that $a^{n-1} + b^{n-1}$ is not a factor of $a^n + b^n$.

SOLUTION Suppose, on the contrary, that $a^{n-1} + b^{n-1}$ is a factor of $a^n + b^n$. That is,

$$a^n + b^n = k(a^{n-1} + b^{n-1}) \qquad (*)$$

for some positive integer k. By symmetry, we may assume that $a \leq b$. Rearranging the previous equation gives

$$b(b^{n-1} - kb^{n-2}) = a^{n-1}(k - a) .$$

Now b is clearly a factor of the left hand side, and therefore also of the right hand side; but b has no common factor with a, so b is a factor of $a - k$. As $b \geq a$ it cannot be that $b \leq a - k$; the only possibility is that $a - k \leq 0$ and $b \leq -(a - k)$. This can be rewritten as $k \geq a + b$. Finally, however, substituting back into $(*)$ yields

$$\text{RHS} \geq (a + b)(a^{n-1} + b^{n-1}) = a^n + ab^{n-1} + a^{n-1}b + b^n > \text{LHS} ;$$

this is impossible, and so $a^{n-1} + b^{n-1}$ cannot, after all, be a factor of $a^n + b^n$.

Q1550 (a) Let $f(x)$ be a polynomial with integral coefficients which has four different integer roots. Prove that there is no integer a such that $f(a)$ is prime.

(b) Find an example of a polynomial $f(x)$ with integral coefficients and three different integer roots, and a value of a such that $f(a)$ is prime.

SOLUTION For (a), if $f(x)$ has four integer roots p, q, r, s, then we can write

$$f(x) = (x - p)(x - q)(x - r)(x - s)g(x) ,$$

where $g(x)$ is a polynomial with integral coefficients. If a is any integer, then $f(a)$ has (at least) four different factors $a - p$, $a - q$, $a - r$ and $a - s$. If $f(a)$ is a prime number, then these factors must be $f(a)$ and 1 and -1 and $-f(a)$; so their product $f(a)^2$ must be a factor of $f(a)$; and this is impossible. So $f(a)$ cannot be prime.

For (b), we'll have

$$f(x) = (x - p)(x - q)(x - r)g(x) .$$

The discussion for (a) indicates that $g(x)$ should "not be there", that is, $g(x) = 1$; and also that for a suitable value of a, two of the factors $a - p$ and $a - q$ and $a - r$ should be 1 and -1. So we may as well take $a = 0$ and $p = 1$ and $q = -1$, so that

$$f(x) = (x - 1)(x + 1)(x - r) , \quad f(a) = r .$$

Now just choose r to be prime, and we are finished. For a specific example we could take

$$f(x) = (x - 1)(x + 1)(x - 101) = x^3 - 101x^2 - x + 101 , \quad a = 0 ,$$

so that $f(a) = 101$ which is prime.